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Basic harmonic forms on a complete Riemannian manifold with non-minimal leaves [1]

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1 Introduction

In this paper, we study the basic harmonic forms on a Riemannian foliation which is non-harmonic. In case of $\mathcal{F}$ is harmonic, many results were obtained([3,5]). If we do not consider the mean curvature of the leaves, many results are similar with the ones in ordinary manifold and then we can consider the results as the generalizations of an ordinary manifold. But if we study the non-harmonic foliation, then the calculations are very difficult. So we need to confine the condition of the mean curvature. An apparent weakening of the condition of the vanishing tension field τ would be to require $\nabla\tau = 0$. But this condition is useless because $\nabla\tau = 0$ implies $\tau = 0$. So we assume that τ is a transversal Killing field and prove the following theorem.

Theorem 1.1 *Let $(M, g_M, \mathcal{F})$ be an Riemannian manifold with complete bundle-like metric g_M and an isoparametric non-minimal Riemannian foliation. Assume that the tension field τ is transverse Killing field. Then every basic L^2-harmonic form ϕ satisfying*

$$liminf \ll \omega_l F(\phi), \omega_l\phi \gg_B \geq 0$$

is parallel.

2 Preliminaries

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{2.1}$$

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determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The transversal Levi-Civita connection ∇ is defined by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.2}$$

where $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$ and ∇^M is the Levi-Civita connection on M. Let R^∇, ρ^∇ and σ^∇ be respectively the curvature tensor, transversal Ricci curvature and the transversal scalar curvature of $\mathcal{F}$. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{2.3}$$

with constant transversal scalar curvature σ^∇. The *mean curvature form* for L is given by

$$\kappa(X) = g_Q(\tau, X) = g_Q(\sum_i \pi(\nabla^M_{E_i} E_i), X) \quad \forall X \in \Gamma Q, \tag{2.4}$$

where $\{E_i\}_{i=1,\cdots,p}$ is a local orthonormal basis of L and τ is a tension field.

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms* $\phi \in \Omega^r(M)$ which satisfy $i(X)\phi = 0$ and $\theta(X)\phi = 0$ for any $X \in \Gamma L$, where $i(X)$ is an interior product and $\theta(X)$ is a Lie derivative. The *basic Laplacian* acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B, \tag{2.5}$$

where δ_B is a formal adjoint of $d_B = d|_{\Omega^*_B(\mathcal{F})}$, which are locally given by

$$d_B = \sum_a E_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa), \tag{2.6}$$

where $\{E_a\}$ is a local orthonormal basic frame on Q and $\kappa = \tau^\sharp$ is a basic mean curvature form.

3 Main results

Now we introduce the operator $\nabla^*_{tr}\nabla_{tr} : \Omega^*_B \to \Omega^*_B$ as

$$\nabla^*_{tr}\nabla_{tr} = -\sum_a \nabla^2_{E_a,E_a} + \nabla_\kappa, \tag{3.1}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in \Gamma TM$. Then we have

Proposition 3.1 *The operator* $\nabla_{tr}^*\nabla_{tr}$ *satisfies*

$$\ll \nabla_{tr}^*\nabla_{tr}\phi_1, \phi_2 \gg_B = \ll \nabla_{tr}\phi_1, \nabla_{tr}\phi_2 \gg_B \tag{3.2}$$

for all $\phi_1,\ \phi_2 \in \Omega_B^*$ *provided that one of* ϕ_1 *and* ϕ_2 *has compact support, where* $\ll \nabla_{tr}\phi_1, \nabla_{tr}\phi_2 \gg_B = \int_M < \nabla_{tr}\phi_1, \nabla_{tr}\phi_2 >_B = \int_M \sum_a < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >$.

Proof. Fix $x \in M$. We choose an orthonormal frame $\{E_a\}$ satisfying $(\nabla E_a)_x = 0$. For any $\phi_1,\ \phi_2 \in \Omega_B^*$,

$$\begin{aligned}
< \nabla_{tr}^*\nabla_{tr}\phi_1, \phi_2 >_B &= -\sum_a < \nabla_{E_a}\nabla_{E_a}\phi_1, \phi_2 >_B + < \nabla_\kappa\phi_1, \phi_2 >_B \\
&= -\sum_a \{E_a < \nabla_{E_a}\phi_1, \phi_2 >_B - < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >_B\} \\
&\quad + < \nabla_\kappa\phi_1, \phi_2 >_B \\
&= -div_\nabla(v) + < \nabla_{tr}\phi_1, \nabla_{tr}\phi_2 >_B + < \nabla_\tau\phi_1, \phi_2 >_B,
\end{aligned}$$

where $v \in \Gamma(Q)$ is defined by the condition that $g_Q(v, w) = < \nabla_w\phi_1, \phi_2 >_B$ for all $w \in \Gamma(Q)$. The last line is proved as follows: At $x \in M$,

$$div_\nabla(v) = \sum_a g_Q(\nabla_{E_a}v, E_a) = \sum_a E_a g_Q(v, E_a) = \sum_a E_a < \nabla_{E_a}\phi_1, \phi_2 >_B .$$

By the Green's theorem on a foliated Riemannian manifold([5]),

$$\int_M div_\nabla(v) = \ll \kappa, v \gg = \ll \nabla_\kappa\phi_1, \phi_2 \gg_B .$$

Hence the proof is completed. □

Now, we define

$$A_\nabla(Y)\phi = \theta(Y)\phi - \nabla_Y\phi \tag{3.3}$$

for any $\phi \in \Omega^*(M)$, $Y \in \Gamma TM$. Since $\nabla_X\phi = \theta(X)\phi$ for $X \in \Gamma L$, $A_\nabla(Y)$ depends only on $s = \pi(Y)$. Moreover, $A_\nabla(Y)$ preserves the basic form up to degree. Hence (3.3) defines

$$A_\nabla(Y) : \Omega_B^r \to \Omega_B^r$$

for any Y. Then we have

Proposition 3.2 (cf.[3]) *Let be a Riemannian foliation. Then the following conditions are equivalent.*

(1) $\pi(Y)$ *is a transverse Killing field, i.e.,* $\theta(Y)g_Q = 0$.

(2) $A_\nabla(Y)$ *is a skew symmetric, i.e.,*

$$< A_\nabla(Y)\phi, \psi >_B + < \phi, A_\nabla(Y)\psi >_B = 0$$

for any $\phi, \psi \in \Omega_B^r$.

Proof. Since ∇ is a metric connection in Ω_B^*, $\nabla_Y < , >_B = 0$. Thus $\theta(Y) < , >_B = 0 \iff A_\nabla(Y) < , >_B = 0$. This implies that $A_\nabla(Y)$ is a skew symmetric. □

Theorem 3.3 *Let $\mathcal{F}$ be the isoparametric foliation. Then*

$$\Delta_B = \nabla_{tr}^* \nabla_{tr} + A_\nabla(\kappa) + F(\phi),$$

where $F(\phi) = \sum_{a,b} \theta_a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$

Proof. Let ϕ be a basic r-form. Let $\{E_a\}$ be an orthonormal basis for Q with $\nabla E_a = 0$ and $\{\theta_a\}$ its dual basic forms. Then we have

$$\begin{aligned}
d_B\delta_B\phi &= \sum_a (\theta_a \wedge \nabla_{E_a})(-\sum_b i(E_b)\nabla_{E_b}\phi + i(\kappa)\phi) \\
&= -\sum_{a,b} \theta_a \wedge \nabla_{E_a}\{i(E_b)\nabla_{E_b}\phi\} + \sum_a \theta_a \wedge \nabla_{E_a} i(\kappa)\phi \\
&= -\sum_{a,b} \theta_a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\kappa)\phi \\
\delta_B d_B\phi &= -\sum_{a,b} i(E_b)\nabla_{E_b}\{\theta^a \wedge \nabla_{E_a}\phi\} + i(\kappa)d_B\phi \\
&= -\sum_{a,b} (i(E_b)\theta^a)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa)d_B\phi + \sum_{a,b} \theta^a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi \\
&= -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} \theta_a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa)d_B\phi
\end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}
\Delta_B\phi &= d_B i(\kappa)\phi + i(\kappa)d_B\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} \theta_a \wedge i(E_b)R^\nabla(E_b, E_a)\phi \\
&= \theta(\kappa)\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} \theta_a \wedge i(E_b)R^\nabla(E_b, E_a)\phi.
\end{aligned}$$

Hence we have

$$\Delta_B\phi = -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \nabla_\kappa\phi + \sum_{a,b}\theta_a \wedge i(E_b)R^\nabla(E_b, E_a)\phi + A_\nabla(\kappa)\phi. \ \square$$

Let x_0 be a fixed point of M. For each point x of M, we denote by $\rho(x)$ the geodesic distance between the leaves passing through x_0 and x, respectively. It is well known that a geodesic orthogonal to leaf is orthogonal to all leaves if g_M is bundle-like metric. Let $B(\ell) = \{y \in M \mid \rho(y) < \ell\}$ for $\ell > 0$. Then there exists a Lipschitz continuous function ω_ℓ on M satisfying the following properties:

$$\begin{aligned} &0 \le \omega_\ell(y) \le 1 \quad \text{for any } y \in M, \\ &\operatorname{supp}\omega_\ell \subset B(2\ell), \\ &\omega_\ell(y) = 1 \quad \text{for any } y \in B(\ell), \\ &\lim_{\ell\to\infty} \omega_\ell = 1, \\ &|d\omega_\ell| \le \frac{C}{\ell} \quad \text{almost everywhere on } M, \end{aligned} \tag{3.4}$$

where $C(>0)$ is a constant independent of ℓ. Then we have

Lemma 3.4 ([4]) *For any $\phi \in \Omega_b^r(\mathcal{F})$, there exists a positive constant A independent of ℓ such that*

$$\|d\omega_\ell \wedge \phi\|^2_{B(2\ell)} \le \frac{A}{\ell^2}\|\phi\|^2_{B(2\ell)},$$
$$\|d\omega_\ell \wedge *\phi\|^2_{B(2\ell)} \le \frac{A}{\ell^2}\|\phi\|^2_{B(2\ell)},$$

where $\|\phi\|^2_{B(2\ell)} = \int_{B(2\ell)} <\phi,\phi>$.

Theorem 3.5 *Let $(M, g_M, \mathcal{F})$ be an Riemannian manifold with complete bundle-like metric g_M and an isoparametric Riemannian foliation. Assume that the tension field τ is transverse Killing field. Then every basic L^2-harmonic forms satisfying*

$$liminf \ll \omega_\ell F(\phi), \omega_\ell\phi \gg_B \ge 0$$

is parallel.

Proof. From Theorem 3.3, we have

$$\ll \Delta_B\phi, \omega_\ell^2\phi \gg_B = \ll \nabla_{tr}^*\nabla_{tr}\phi + A_\nabla(\kappa)\phi + F(\phi), \omega_\ell^2\phi \gg_B \tag{3.5}$$

Since κ is a transverse Killing field, from Proposition 3.2,

$$< A_\nabla(\tau)\phi, \phi >_B = 0.$$

Also, by using Lemma 3.4 and the inequality $| < a, b > | \le \frac{1}{t}|a|^2 + t|b|^2$ for any positive real number t, we have

$$\begin{aligned} \ll \nabla_{tr}^* \nabla_{tr}\phi, \omega_\ell^2 \phi \gg_B &= \ll \nabla_{tr}\phi, 2\omega_\ell d_B \omega_\ell \wedge \phi \gg_B + \|\omega_\ell \nabla_{tr}\phi\|_B^2 \\ &\ge \frac{1}{2}\|\omega_\ell \nabla_{tr}\phi\|_B^2 - \frac{4A}{\ell^2}\|\phi\|_B^2. \end{aligned}$$

From (3.5), for any basic L^2-harmonic r form ϕ, letting $\ell \to \infty$, we get

$$\frac{1}{2}\|\nabla_{tr}\phi\|_B^2 + liminf \ll \omega_\ell F(\phi), \omega_\ell \phi \gg_B \le 0.$$

Under our assumption, the proof is completed. □

Now we calculate $< F(\phi), \phi >_B$ precisely. Let ϕ be a basic 1-form and ϕ^* its g_Q-dual. Then

$$\begin{aligned} < F(\phi), \phi >_B &= \sum_{a,b} < \theta^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi, \phi >_B \\ &= \sum_{a,b} i(E_b) R^\nabla(E_b, E_a)\phi < \theta^a, \phi >_B \\ &= \sum_{a,b} R^\nabla(E_b, E_a)\phi^b < \theta^a, \phi >_B \\ &= \rho^\nabla(\phi^*, \phi^*), \end{aligned}$$

where ρ^∇ is transversal Ricci curvature. From Theorem 3.5, we have

Corollary 3.6 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with complete bundle-like metric g_M and an isoparametric Riemannian foliation $\mathcal{F}$. Assume that the mean curvature form κ is a transverse Killing field. If the transversal Ricci curvature is non-negative, then every basic L^2-harmonic 1-form is parallel. If the transversal Ricci curvature is quasi positive, then every basic L^2-harmonic 1-form is zero.*

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