科學敎育, 11, 47~66, 1994
Journal of Science Education
11, 47~66, 1994

전위에 의한 정칙곡면의 넓이

문영봉* · 현진오**

The Area of Regular Surfaces Under Inversion

Moon, Young-Bong* · Hyen, Jin-Oh**

Abstract

A mapping $f : E^3 - \{(0,0,0)\} \to E^3$ which sends a point p into a point p' is called an inversion in an Euclidean space E^3 with respect to a given circle or sphere which center O and radius R, if $OP \cdot OP' = R^2$ and if the points P, P' are on the same side of O and O, P, P' are collinear.

This thesis shows that, a bounded region M of a regular surface S in E^3 and a parametrization $X(u,v) = (x(u,v), y(u,v), z(u,v))$ of S being given, the area of $f(M)$ under inversion is equal to $\iint_Q \frac{1}{|X|^4}\sqrt{EG-F^2}\,du\,dv$, where $Q = X^{-1}(M)$.

Introduction

In this paper, our study of area will be restricted to the regular surface in the Euclidean space E^3.

In Section 1, we present the basic concepts of a regular surface in E^3

* 대정고등학교 교사
** 제주대학교 사범대학 수학교육과

and introduce the first fundamental form, a natural instrument to treat the area of region on a regular surface. And we also show how to find the area of a regular surface.

Next, in Section 2, we introduce the definition and some properties of inversion in E^3 and show that an inversion $f : S \to \bar{S}$ of two regular surfaces $S, \bar{S}$ in E^3 is a local conformal mapping. That is, the first fundamental forms of $S, \bar{S}$ are proportional.

Finally, in Section 3, we present the main theorem ; the area $f(M)$ of a bounded region M of a regular surface S under an inversion $f : S \to \bar{S}$ is equal to $R^4 \iint\limits_Q \frac{1}{|X|^4} \sqrt{EG - F^2}\, du\, dv$, where $Q = X^{-1}(M)$.

1. The area of a regular surface

We shall introduce the basic concept of regular surface in E^3. Regular surfaces are defined as sets rather than maps. A regular surface in E^3 is a subset of E^3.

Definition 1.1. A subset $S \subset E^3$ is a regular surface if, for each $p \in S$ there exits a neighborhood V of p in E^3 and a map $X : U \to V \cap S$ of an open set $U \subset E^2$ onto $V \cap S \subset E^3$ subject to the following three conditions:

(i) X is differentiable.

(ii) X is a homeomorphism.

(iii) For each $q \in U$, the differential $dX_q : E^2 \to E^3$ is one-to-one.

If we write $X(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in U$, then the functions $x(u,v)$, $y(u,v), z(u,v)$ have continuous partial derivatives of all orders in U. Since X is continuous by condition (i), condition (ii) means that X has an inverse $X^{-1} : V \cap S \to U$ which is continuous. Let us compute the matrix of the linear map dX_q in the canonical bases $e_1 = (1,0), e_2 = (0,1)$ of E^2 with coordinates (u,v) and $i_1 = (1,0,0), i_2 = (0,1,0), i_3 = (0,0,1)$ of E^3, with coordinates (x,y,z). Then, by the definition of differential,

$$dX_q(e_1) = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) = \frac{\partial X}{\partial u} = X_u, \tag{1.1}$$

$$dX_q(e_2) = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) = \frac{\partial X}{\partial v} = X_v. \tag{1.2}$$

Condition (iii) means that the Jacobian matrix $J_x(q)$ of the mapping X at each $q \in U$ has rank 2. This implies that at each $q \in U$ the vector product $\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \neq O$ (regularity condition), where $(u,v) \in U$. Thus the regular surface S is neither a point nor a curve.

The mapping X is called a parametrization or a system of local coordinates in a neighborhood of p. The neighborhood $V \cap S$ of $p \in S$ is called a coordinate neighborhood.

Example 1.2. Let the sphere $S^2 = \{(x,y,z) \in E^3; x^2+y^2+z^2 = a^2\}$. Consider the map $X_1 : U = \{(x,y) \in E^2; x^2+y^2 < a^2\} \to S^2_+$ given by $X_1(x,y) = \left(x, y, \sqrt{a^2-(x^2+y^2)}\right)$, where $S^2_+ = \{(x,y,z) \in S^2; z > 0\}$.

Since $x^2+y^2 < a^2$, the function $f_3(x,y) = \sqrt{a^2-(x^2+y^2)}$ has continuous partial derivatives of all orders. Thus condition (i) holds. Since X_1 is one-to-one, and X_1^{-1} is the restriction of the projection : $(x,y,z) \to (x,y,0)$, X_1^{-1} is continuous and satifies condition ii). Condition iii) is easily verified, since the Jacobian matrix $\begin{pmatrix} 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{pmatrix}$ of the map X_1 at each $q \in U$ has rank 2. Thus the map X_1 is a parametrization of S^2.

Similarly, we have the parametrizations

$$X_2(x,y) = \left(x, y, -\sqrt{a^2-(x^2+y^2)}\right),$$

$$X_3(x,z) = \left(x, \sqrt{a^2-(x^2+Z^2)}, z\right),$$

$$X_4(x,z) = \left(x, -\sqrt{a^2-(x^2+z^2)}, z\right),$$

$$X_5(y,z) = \left(\sqrt{a^2-(y^2+z^2)}, y, z\right),$$

$$X_6(y,z) = \left(-\sqrt{a^2-(y^2+z^2)}, y, z\right),$$

which, together with X_1, cover S^2 completely, and show that S^2 is a regular surface.

Definition 1.3. The tangent space of a regular surface S at $p \in S$ is the set $T_p(S)$ of all vectors tangent to S at p.

Definition 1.4. The quadratic form I_p on $T_p(S)$, defined by $I_p(\boldsymbol{w}) = <\boldsymbol{w}, \boldsymbol{w}>_p = |\boldsymbol{w}|^2 \geq 0$, is called the first fundamental form of the regular surface $S \subset E^3$ at $p \in S$, where $\boldsymbol{w} \in T_p(S)$.

We shall now express the first fundamental form in the basis $\{X_u, X_v\}$ associated to a parametrization $X(u,v)$ at p. Since a tangent vector $\boldsymbol{w} \in T_p(S)$ is the tangent vector to a parametrized curve $\alpha(t) = X(u(t), v(t))$, $t \in (-\varepsilon, \varepsilon)$, with $p = \alpha(0) = X(u_0, v_0)$, we obtain

$$\begin{aligned} I_p(\alpha'(0)) &= <\alpha'(0), \alpha'(0)>_p \\ &= <X_u u' + X_v v', X_u u' + X_v v'>_p \\ &= <X_u, X_u>_p (u')^2 + 2 <X_u, X_v>_p u'v' + <X_v, X_v>_p (v')^2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2, \end{aligned}$$

where

$$E(u_0, v_0) = <X_u, X_u>_p, \tag{1.3}$$

$$F(u_0, v_0) = <X_u, X_v>_p, \tag{1.4}$$

$$G(u_0, v_0) = < X_v, X_v >_p, \tag{1.5}$$

are the coefficients of the first fundamental form in the basis $\{X_u, X_v\}$ of $T_p(S)$. By letting p run in the coordinate neighborhood corresponding to $X(u,v)$ we obtain functions $E(u,v), F(u,v), G(u,v)$ which are differentiable in that neighborhood.

Definition 1.5. Let $M \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood corresponding to the parametrization $X : U \subset E^2 \to S$. The positive number

$$\iint_Q \mid X_u \times X_v \mid du\, dv = A(M), \quad Q = X^{-1}(M), \tag{1.6}$$

is called the area of M.

The function $\mid X_u \times X_v \mid$, defined in U, measures the area of the parallelogram generated by the vectors X_u and X_v.

Proposition 1.6. In the coordinate neighborhood corresponding to the parametrization $X(u,v)$,

$$A(M) = \iint_Q \sqrt{EG - F^2}\, du\, dv, \quad Q = X^{-1}(M). \tag{1.7}$$

Proof. Let θ be the angle between X_u and X_v. Then

$$\mid X_u \times X_v \mid^2 = |X_u|^2 |X_v{}^2 \sin^2 \theta$$

$$= |X_u|^2|X_v|^2(1-\cos^2\theta)$$
$$= |X_u|^2|X_v|^2\left(1-\frac{<X_u,X_v>^2}{|X_u|^2|X_v|^2}\right)$$
$$= |X_u|^2|X_v|^2 - <X_u,X_v>^2$$
$$= EG-F^2.$$

Corollary 1.7. The parametrization $X(u,v)$ has the regularity condition if and only if $EG-F^2$ is never zero, that is, $EG-F^2>0$.

Example 1.8. Let S be a sphere with radius r and center O and let $U=\left\{(u,v)\in E^2; 0<u<2\pi, -\frac{\pi}{2}<v<\frac{\pi}{2}\right\}$. If $X:U\to E^3$ is given by $X(u,v)=(r\cos v\cos u, r\cos v\sin u, r\sin v)$, then

$$E=r^2\cos^2 v, F=0, G=r^2.$$

Now, consider the region S_ϵ obtained as the image by X of the region Q_ϵ given by $Q_\epsilon=\left\{(u,v); 0+\varepsilon\le u\le 2\pi-\varepsilon, -\frac{\pi}{2}-\varepsilon\le v\le\frac{\pi}{2}+\varepsilon\right\}$, $\varepsilon>0$.

Using (1.4), we obtain

$$A(S_\epsilon)=\int_{0+\epsilon}^{2\pi-\epsilon}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon}\sqrt{EG-F^2}\,dv\,du$$
$$=\int_{0+\epsilon}^{2\pi-\epsilon}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon} r^2\cos v\,dv\,du$$
$$=4r^2(\pi-\varepsilon)\cos\varepsilon.$$

Letting $\varepsilon \to 0$,

$$A(S) = 4\pi r^2.$$

2. The conformal map of two regular surfaces under inversion

Let the symbol $(O)_R$ denote the circle (sphere) with center O and radius R.

Definition 2.1. Two points P and P' of $E^2(E^3)$ are said to be inverse with respect to a given $(O)_R$, if

$$OP \cdot OP' = R^2 \tag{2.1}$$

and if P, P' are on the same side of O and the points O, P, P' are collinear.

A $(O)_R$ is called the circle(sphere) of inversion, and the transformation which sends a point P into P' is called an inversion.

The center O of the circle(sphere) of inversion has no inverse point.

The center O put the origin in the coordinate system. Denote the distance to the origin O of a point $X \in E^3$ by $|X|$.

Proposition 2.2. An inversion in a space E^3 is a mapping $f : E^3 -$

$\{(0,0,0)\} \to E^3$ such that

$$f(X) = \frac{R^2 X}{< X, X >} = \frac{R^2 X}{|X|^2}. \tag{2.2}$$

Proof. For some positive real number k, $f(X) = kX$,

because the points O, P, P' are collinear.

Since $f(X)$ is the inverse of X, by means of (2.1),

$$| X || f(X) |= R^2,$$

$$k | X |^2 = R^2.$$

Since $| X | \neq 0$, $$k = \frac{R^2}{| X |^2}.$$

Hence (2.2) holds.

The inversion $f(X) = \dfrac{R^2 X}{| X |^2}$ is the vector of length $R^2 | X |^{-1}$ on the ray of X, and is not defined for $X = O$ nor is $Y = O$ the image point of any $X \in E^3$.

Proposition 2.3.

(1) A line through O inverts into a line through O.

(2) A line not through O inverts into a circle through O.

(3) A circle through O inverts into a line not through O.

(4) A circle not through O inverts into a circle not through O.

When the words line and circle are interchanged with the words plane and sphere, respectively, Proposition 2.3 is stated in the next Theorem 2.4.

Theorem 2.4.

(1) A plane through O inverts into a plane through O.

(2) A plane not through O inverse into a sphere through O.

(3) A sphere through O inverts into a plane not through O.

(4) A sphere not through O inverts into a sphere not through O.

Proof. Let B be any vector in E^3 and consider the equation

(2.3) $\quad a|X|^2 + \langle B, X \rangle + c = 0$, where a, c are real numbers.

Then the equation (2.3) represents a sphere for $a \neq 0, c \neq 0$, and a plane for $a = 0, B \neq O$.

For $|X| \neq 0$, multiplying both sides of (2.3) by $\frac{R^2}{|X|^2}$,

(2.4.a)
$$R^2 a + \frac{R^2 \langle B, X \rangle}{|X|^2} + \frac{R^2 c}{|X|^2} = 0.$$

Let $Y = \frac{R^2 X}{|X|^2}$. Then

(2.4.b)
$$\frac{c}{R^2}|Y|^2 + \langle B, Y \rangle + R^2 a = 0.$$

Thus (2.3) under inversion is transformed into (2.4.b).

(1) When $a = 0, B \neq O, c = 0$, (2.3) and (2.4.b) represent a plane through O.

(2) When $a = 0, B \neq O, c \neq 0$, (2.3) represents a plane not through O and (2.4.b) represents a sphere through O.

(3) When $a \neq 0, B \neq O, c = 0$, (2.3) represents a sphere through O and (2.4.b) represents a plane not through O.

(4) When $a \neq 0, B \neq O, c \neq 0$, (2.3) and (2.4.$b$) represent a sphere not through O.

Definition 2.5. A conformal mapping $f : S \to \bar{S}$ of two regular surfaces $S, \bar{S}$ in E^3 is a bijective differentiable mapping that preserves the angle between any two intersecting curves on the regular surface S.

A mapping $f : V \to \bar{S}$ of a neighborhood V of a point p on a regular surface S into $\bar{S}$ is a local conformal mapping at p if there exists a neighborhood $\bar{V}$ of $f(p) \in \bar{S}$ such that $f : V \to \bar{V}$ is a conformal mapping. If there exists a local conformal mapping at each $p \in S$, the regular surface S is locally conformal to the regular surface $\bar{S}$.

Theorem 2.6. A mapping $f : S \to \bar{S}$ of two regular surfaces $S, \bar{S}$ is a local conformal mapping at $p \in S$ if the first fundamental forms of $S, \bar{S}$ at $p, f(p)$, respectively, are proportional, that is, $\bar{E} = \lambda^2 E, \bar{F} = \lambda^2 F, \bar{G} = \lambda^2 G$,

$\lambda(u,v) > 0$.

Proof. Let $X(u,v)$ be a parametrization of the regular surface S, and $f(X(u,v)) = \bar{X}(u,v)$ be that of $\bar{S}$. Let C_1, C_2 be two curves on the regular surface S intersecting at a point $p = X(u,v)$ given by the coordinate functions, respectively,

$$u = u_1(s_1), v = v_1(s_1); u = u_2(s_2), v = v_2(s_2), \tag{2.5}$$

where s_1, s_2 are the arc length of C_1, C_2.

Then the unit tangent vectors of C_1, C_2 at p are, respectively,

$$\mathbf{t}_1 = X_u \frac{du_1}{ds_1} + X_v \frac{dv_1}{ds_1}, \tag{2.6}$$

$$\mathbf{t}_2 = X_u \frac{du_2}{ds_2} + X_v \frac{dv_2}{ds_2}. \tag{2.7}$$

From (2.5) the angle θ between $\mathbf{t}_1, \mathbf{t}_2$ is therefore given by

$$\begin{aligned} \cos\theta &= \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \\ &= \frac{1}{ds_1 ds_2}[E du_1 du_2 + F(du_1 dv_2 + du_2 dv_1) + G dv_1 dv_2], \end{aligned} \tag{2.8}$$

provided that the sign of $\sin\theta$ is properly chosen. Thus we have

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ &= 1 - \frac{1}{ds_1^2 ds_2^2}[E du_1 du_2 + F(du_1 dv_2 + du_2 dv_1) + G dv_1 dv_2]^2 \\ &= \frac{1}{ds_1^2 ds_2^2}(EG - F^2)(du_1 dv_2 - du_2 dv_1)^2, \end{aligned}$$

where

$$ds_1^2 = Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2,$$

$$ds_2^2 = Edu_2^2 + 2Fdu_2dv_2 + Gd_2^2.$$

Let $\bar{\theta}$ be the angle between the curves corresponding to $\bar{C}_1, \bar{C}_2$ under f at the corresponding point $f(p)$ on the surface $\bar{S}$. Then by replacing E, F, G, respectively, by $\bar{E}, \bar{F}, \bar{G}$, the coefficients of the first fundamental form on $\bar{S}$, using

(2.9) $$\sin\theta = \frac{\sqrt{EG - F^2}}{ds_1 ds_2}(du_1dv_2 - du_2dv_1),$$

and putting $\bar{E} = \lambda^2 E, \bar{F} = \lambda^2 F, \bar{G} = \lambda^2 G$, where λ^2 is an arbitrary nonzero function of u, v, and the positive square root is to be taken for λ, we have

$$\begin{aligned}\cos\bar{\theta} &= \frac{1}{d\bar{s}_1^2 d\bar{s}_2^2}[\bar{E}du_1du_2 + \bar{F}(du_1dv_2 + du_2dv_1) + \bar{G}dv_1dv_2] \\ &= \frac{1}{\lambda^2 ds_1 ds_2}\lambda^2[Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2] \\ &= \cos\theta,\end{aligned}$$

$$\begin{aligned}\sin\bar{\theta} &= \frac{1}{d\bar{s}_1 d\bar{s}_2}\sqrt{\bar{E}\bar{G} - \bar{F}^2}(du_1dv_2 - du_2dv_1) \\ &= \frac{1}{\lambda^2 ds_1 ds_2}\lambda^2\sqrt{EG - F^2}(du_1dv_2 - du_2dv_1) \\ &= \sin\theta,\end{aligned}$$

where

$$d\bar{s}_1^2 = \bar{E}du_1^2 + 2\bar{F}du_1dv_1 + \bar{G}dv_1^2,$$

$$d\bar{s}_2^2 = \bar{E}du_2^2 + 2\bar{F}du_2dv_2 + \bar{G}dv_2^2.$$

Thus $\bar{\theta} = \theta$, and f is a local conformal mapping.

Theorem 2.7. An inversion $f : S \to \bar{S}$ is a local conformal mapping of two regular surfaces, that is, S is locally conformal to $\bar{S}$.

Proof. Let E, F, G and $\bar{E}, \bar{F}, \bar{G}$ be, respectively, the coefficients of the first fundamental form of a regular surface S and its image regular surface $\bar{S} = f(S)$.

By using of (2.2),

$$\begin{aligned}\frac{\partial f(X)}{\partial u} &= R^2\frac{X_u < X, X > -X(< X_u, X > + < X, X_u >)}{< X, X >^2}\\ &= R^2\frac{X_u < X, X > -2X < X_u, X >}{< X, X >^2},\end{aligned}$$

$$\frac{\partial f(X)}{\partial v} = R^2\frac{X_v < X, X > -2X < X_v, X >}{< X, X >^2},$$

$$\bar{E} = < \frac{\partial f(X)}{\partial u}, \frac{\partial f(X)}{\partial u} > = \frac{R^4}{|X|^4}E, \tag{2.10}$$

$$\bar{F} = \frac{R^4}{|X|^4}F, \tag{2.11}$$

$$\bar{G} = \frac{R^4}{|X|^4}G. \tag{2.12}$$

The first fundamental forms of $S, \bar{S}$ are proportional. Thus the regular surface S is locally conformal to the regular surface $\bar{S}$.

Remark. In the Theorem 2.7, if $EG - F^2 > 0$, then $\bar{E}\bar{G} - \bar{F}^2 > 0$. In an inversion $f : S \to \bar{S}$, two surfaces $S, \bar{S}$ are regular.

3. The area under inversion

Theorem 3.1. Let $M \subset S$ be the bounded region of a regular surface S in $E^3 - \{(0,0,0)\}$ and let $X : U \to S$ be a map given by $X(u,v) = (x(u,v), y(u,v), z(u,v))$. If the mapping $f : S \to \bar{S}$ is an inversion,then the area of $f(M)$ is equal to

$$R^4 \iint_Q \frac{1}{|X|^4} \sqrt{EG - F^2}\, du\, dv, \tag{3.1}$$

where $Q = X^{-1}(M) = \{(u,v);\ u_1 \le u \le u_2, v_1 \le v \le v_2\}$.

Proof. Let $\bar{E}du^2 + 2\bar{F}du\,dv + \bar{G}dv^2$ be the first fundamental form of an image surface $\bar{S} = f(S)$. Then, by using of (2.10), (2.11), (2.12), the area of $f(M)$ is given by

$$\begin{aligned} \iint_Q \sqrt{\bar{E}\bar{G} - \bar{F}^2}\, du\, dv &= \iint_Q \frac{R^4}{|X|^4} \sqrt{EG - F^2}\, du\, dv \\ &= R^4 \iint_Q \frac{1}{|X|^4} \sqrt{EG - F^2}\, du\, dv. \end{aligned}$$

Example 3.2. Let $S = \{(x, y, z) \in E^3;\ z = 0, (x, y) \in V : \text{open set}\}$ be the xy plane and let $X : U \to S$ be a parametrization of S given by

$$X(u, v) = (2u\cos^2 v, 2u\cos v\sin v, 0),$$

where $U = \left\{(u, v) \in E^2; 0 < u, -\frac{\pi}{2} < v < \frac{\pi}{2}\right\}$. Then

$E = 4\cos^2 v, \quad F = -4u\cos v\sin v, \quad G = 4u^2, \quad |X|^4 = 16u^4\cos^4 v.$

If $Q = \left\{(u, v); \frac{1}{2} \le u \le 2, 0 \le v \le \frac{\pi}{6}\right\}$, then

$$\begin{aligned} A(f(M)) &= R^4 \int_0^{\frac{\pi}{6}} \int_{\frac{1}{2}}^{2} \frac{1}{|X|^4}\sqrt{EG - F^2}\, du\, dv \\ &= R^4 \int_0^{\frac{\pi}{6}} \int_{\frac{1}{2}}^{2} \frac{1}{4u^3\cos^2 v}\, du\, dv \\ &= \frac{5\sqrt{3}}{32} R^4. \end{aligned}$$

On the other hand, if $C_1' : \beta_1(t) = \dfrac{R^2}{4\cos t}$, $C_2' : \beta_2(t) = \dfrac{R^2}{\cos t}$, as shown in < Fig. 3.1 >, then

$$\begin{aligned} A(f(M)) &= \int_0^{\frac{\pi}{6}} \int_{\beta_1(t)}^{\beta_2(t)} r\, dr\, dt \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} [\beta_2^2(t) - \beta_1^2(t)]\, dt \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{15}{16} R^4 \sec^2 t\, dt \\ &= \frac{15R^4}{32} \tan t \Big|_0^{\frac{\pi}{6}} \\ &= \frac{5\sqrt{3}}{32} R^4. \end{aligned}$$

Example 3.3. Let $S = \{(x, y, z); x^2 + y^2 + (z - 2)^2 = 1\}$ and let

$X : U \to S$ be a parametrization of a regular surface S given by

$$X(u,v) = (\cos u \cos v, \sin u \cos v, \sin v + 2),$$

where $U = \left\{(u,v) | 0 < u < 2\pi, -\frac{\pi}{2} < v < \frac{\pi}{2}\right\}$. Then

$$E = \cos^2 v, \quad F = 0, \quad G = 1, \quad |X|^4 = (5 + 4\sin v)^2.$$

Consider the region $f(M)_\varepsilon$ obtained as the image by $f(X)$ of the region Q_ε given by $Q_\varepsilon = \left\{(u,v) \in E^2; 0 + \varepsilon \le u \le 2\pi - \varepsilon, -\frac{\pi}{2} + \varepsilon \le v \le \frac{\pi}{2} - \varepsilon\right\}$ as shown in < Fig. 3.2 > .

The area of $f(M)_\varepsilon$ is

$$\begin{aligned} A(f(M)_\varepsilon) &= \int_\varepsilon^{2\pi-\varepsilon} \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} \sqrt{\frac{R^8 \cos^2 v}{(5+4\sin v)^4}}\, dv\, du \\ &= R^4 \int_\varepsilon^{2\pi-\varepsilon} \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}+\varepsilon} \frac{\cos v}{(5+4\sin v)^2}\, dv\, du \\ &= \frac{R^4}{4}\{(5 - 4\cos\varepsilon)^{-1} - (5 + 4\cos\varepsilon)^{-1}\}(2\pi - 2\varepsilon). \end{aligned}$$

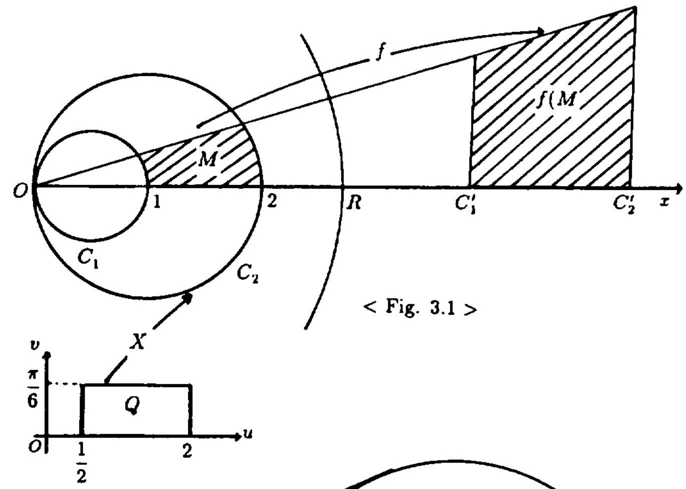
Letting $\varepsilon \to 0$,

$$\begin{aligned} A(f(M)) &= \frac{R^4}{4}\left(2\pi - \frac{2\pi}{9}\right) \\ &= \frac{4}{9}\pi R^4. \end{aligned}$$

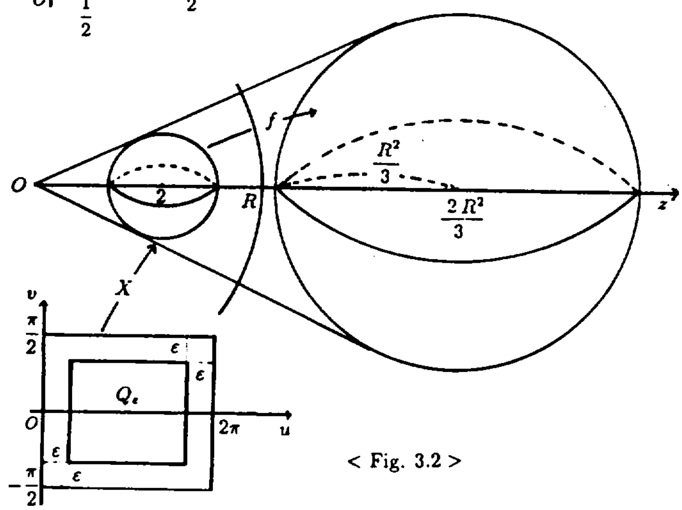
On the other hand, in virtue of (2.3) and (2.4.b), if $a = 1$, $B = (0,0,-4)$, $c = 3$, then $S = \{(x,y,z);\ x^2 + y^2 + (z-2)^2 = 1\}$ is transformed into $\bar{S} = \left\{(x,y,z); x^2 + y^2 + \left(z - \frac{2R^2}{3}\right)^2 = \frac{R^4}{9}\right\}$.

Thus

$$A(f(M)) = \frac{4}{9}\pi R^4.$$



< Fig. 3.1 >



< Fig. 3.2 >

REFERENCES

[1] 양창홍 (1993), "On the arc lengh under inversion", 석사학위논문, 제주대학교 교육대학원.

[2] Mandredo P.Do Carmo(1976), *Differential Geometry of Curves and Surfaces*, Prentice-Hall.

[3] Chuan-Chih Hsiung(1981), *A First Course in Differential Geometry,* John Wiley & Sons.

[4] Richard S.Millman and George D.Parker(1977), *Elements of Differential Geometry,* Prentice-Hall.

[5] Heinrich W.Guggenheimer(1963), *Differential Geometry,* Mcgraw-Hill.

[6] Barrett O'Neill(1966), *Elementary Differential Geometry,* Academic Press.

[7] Claire Fisher Adler(1967), *Modern Geometry,* McGraw-Hill.

[8] R.Creghton Buck(1978), *Advanced Calculus,* McGraw-Hill.

〈초 록〉

전위에 의한 정칙곡면의 넓이

중심이 O이고 반지름의 길이가 R인 주어진 원 또는 구에서 Euclid 공간 E^3의 두점 P, P' 이 중심 O의 같은 쪽에 있고 $OP \cdot OP' = R^2$, 점 O, P, P' 이 동일 직선상에 있을 때 점 P 에서 P' 으로 보내는 변환 $f : E^3 - \{(0,0,0)\} \to E^3$ 를 전위라 한다.

이 논문은 Euclid 공간 E^3 에서 정치곡면 S의 유계영역이 M일때 S의 좌표근방 $X(U)$의 국소표현 $X(u,v) = (x(u,v), y(u,v), z(u,v))$ 가 주어지면 전위 f에 의한 $f(M)$의 넓이는 $\iint_Q \frac{1}{|X|^4}\sqrt{EG-F^2}\,du\,dv$, (단, $Q = X^{-1}(M)$) 임을 보인다.