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TERM RANK PRESERVERS BETWEEN DIFFERENT GENERAL BOOLEAN MATRICES

濟州大學校　大學院

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TERM RANK PRESERVERS BETWEEN DIFFERENT GENERAL BOOLEAN MATRICES

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CONTENTS

<**Abstract**>

TERM RANK PRESERVERS BETWEEN DIFFERENT GENERAL BOOLEAN MATRIX SPACES

In this thesis we consider linear transformations from $m \times n$ general Boolean matrices into $p \times q$ general Boolean matrices that preserve term rank. We study linear transformation that preserve term rank between different general Boolean matrix spaces. This results extend the results on the linear operators from $m \times n$ binary Boolean matrices into itself that preserve term rank.

The term rank of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k-h$ columns. The term rank of a matrix A is denoted by $\tau(A)$.

Let $\Xi_k^{(r,s)}$ denote the set of all matrices in $\mathbb{M}_{r,s}(\mathbb{B}_k)$ whose term rank is k.

Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \rightarrow \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation from $m \times n$ general Boolean matrices into $p \times q$ general Boolean matrices. If f is a function defined on matrix spaces, then T preserves the function f if $f(T(A)) = f(A)$ for all A in a matrix space. If $\mathbb{X}$ is a subset of a matrix space and $\mathbb{Y}$ is a subset of a matrix space, then we say T preserves the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. And, we also say T strongly preserves the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ if and only if $T(A) \in \mathbb{Y}$. Further, we say that T (strongly) preserves term rank k if T (strongly) preserves the pair $(\Xi_k^{(m,n)}, \Xi_k^{(p,q)})$.

Song and Beasley characterized linear transformation that preserve term rank on an antinegative semirings without zero-divisor. But in this paper, we characterize linear transformations that preserve term rank between different general Boolean matrix spaces with zero-divisor.

The main results are the following:

Theorem. Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \rightarrow \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation. Then the following are equivalent:

1. T preserves term rank;
2. T preserves term rank k and term rank l, with $1 \leq k < l \leq m \leq n$ and $k+1 < m$;
3. T strongly preserves term rank h, with $1 \leq h < m \leq n$;
4. T has the form $T(X) = P(X \oplus O)Q$, where P, Q are permutation matrices of order p and q, respect.

1 Introduction

There are many papers on linear operators on a matrix space that preserve matrix functions over various algebraic structures ([8]). But there are few papers of linear transformations between another matrix spaces that preserve matrix functions over an algebraic structure([11]).

Let $\mathbb{F}$ be a field and $\mathbb{M}_{m,n}(\mathbb{F})$ denote the vector space of all $m \times n$ matrices over $\mathbb{F}$. Over the last century, a great deal of effort has been devoted to the following problem. Characterize those linear operators $T : \mathbb{M}_{m,n}(\mathbb{F}) \to \mathbb{M}_{m,n}(\mathbb{F})$ which leave a function or set invariant. We call this a Linear Preserver Problem(LPP) ([7], [8]). There are four general types of LPP. The most typical and oldest type of LPP is as follows ([10]):

I. Let f be a function on $\mathbb{M}_{m,n}(\mathbb{F})$. Characterize those T on $\mathbb{M}_{m,n}(\mathbb{F})$ such that $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{F})$.

The study of these operators began in 1897 ([8]) when Frobenius characterized the linear operators that preserve the determinant over complex matrices and over real symmetric matrices.

Another types of LPP is as follows:
II-1. Let $S \subset \mathbb{M}_{m,n}(\mathbb{F})$. Characterize those T on $\mathbb{M}_{m,n}(\mathbb{F})$ such that $T(S) \subset S$ or $T(S) = S$.

II-2. Let $_r$ be a relation(or an equivalent relation) over $\mathbb{M}_{m,n}(\mathbb{F})$. Characterize those T on $\mathbb{M}_{m,n}(\mathbb{F})$ such that A_rB if and only if $T(A)_rT(B)$.

II-3. Let f be a transformation from $\mathbb{M}_{m,n}(\mathbb{F})$ to $\mathbb{M}_{m,n}(\mathbb{F})$. Characterize those T on $\mathbb{M}_{m,n}(\mathbb{F})$ such that $f(T(A)) = T(f(A))$ for all $A \in \mathbb{M}_{m,n}(\mathbb{F})$.

We now turn our attention to matrices over semirings, in particular Boolean algebra.

Boolean algebra is named after the British Mathematician George Boole (1813 - 1864). The Boolean algebra of two elements is most frequently used in combinatorial applications, and all other finite Boolean algebras are direct sums of copies of it ([6]).

Applications of the theory of Boolean matrices are of fundamental importance in the formation and analysis of many classes of discrete structural models which arise in the physical, biological, and social sciences. The theory is also intimately related to many branches of mathematics, including relation theory, logic, graph theory, lattice theory and algebraic semigroup theory ([4], [6]).

The study of the characterization of linear operators that preserve invariants of matrices over semirings is a counterpart for the study of preservers over fields, and it has its own importance.

In [1], Beasley and Pullman established analogous results over Boolean algebra to many preserver problems for matrices over field.

At the most recent, studying for characterization of linear operator(and linear transformation) that preserve term rank performed by Beasley, Song and Kang([3], [5], [11]).

In this paper we consider linear transformations from $m \times n$ general Boolean matrices into $p \times q$ general Boolean matrices that preserve term rank. We study linear transformation that preserve term rank between different general Boolean matrix spaces. This results extend the results on the linear operators from $m \times n$ binary Boolean matrices into itself that preserve term rank.

2 Preliminaries and Definitions

Definition 2.1. *A semiring is a set* $\mathbb{S}$ *equipped with two binary operations* $+$ *and* $\cdot$ *such that* $(\mathbb{S}, +)$ *is a commutative monoid with identity element* 0 *and* $(\mathbb{S}, \cdot)$ *is a monoid with identity element* 1*. In addition, the operations* $+$ *and* $\cdot$ *are connected by distributivity of* $\cdot$ *over* $+$*, and* 0 *annihilates* $\mathbb{S}$*.*

Definition 2.2. *A semiring* $\mathbb{S}$ *is called antinegative if* 0 *is the only element to have an additive inverse.*

The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B} = \{0, 1\}$. Then $(\mathbb{B}, +, \cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1 + 1 = 1$. If $\mathbb{P}$ is any subring of $\mathbb{R}$ with identity, the reals (under real addition and multiplication), and $\mathbb{P}^+$ denotes the nonnegative part of $\mathbb{P}$, then $\mathbb{P}^+$ is an antinegative semiring. In particular $\mathbb{Z}^+$, the nonnegative integers, is an antinegative semiring.

Definition 2.3. *Let* $\mathbb{B}_k = P(\{1, 2, \cdots, k\})$*. (*$P(A)$ *means the power set of set A.) Union is denoted by + and intersection by* $\cdot$ *(or juxataposition); 0 denotes the null set and 1 the set*$\{1, 2, \cdots, k\}$*. Then* $(\mathbb{B}_k, +, \cdot)$ *is an antinegative semiring (the general Boolean semiring([12])). In particular, if* $k = 1$*,* $\mathbb{B}_1$ *is a binary Boolean semiring.*

Definition 2.4. *A nonzero* $s \in \mathbb{S}$ *is a zero-divisor if* $s's = 0$ *for some nonzero* $s' \in \mathbb{S}$*. The binary Boolean semiring is an antinegative semiring without zero-divisor, but the general Boolean semiring is not.*

Hereafter, $\mathbb{S}$ will denote an arbitrary commutative and antinegative semiring.

Definition 2.5. *Let* $\mathbb{M}_{m,n}(\mathbb{S})$ *and* $\mathbb{M}_{p,q}(\mathbb{S})$ *be the set of all* $m \times n$ *and* $p \times q$ *matrices respectively with entries in a semiring* $\mathbb{S}$*. Algebraic operations on* $\mathbb{M}_{m,n}(\mathbb{S})$ *and* $\mathbb{M}_{p,q}(\mathbb{S})$ *are defined as if the underlying scalars were in a field.*

Definition 2.6. *The term rank, of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k-h$ columns. The term rank of a matrix A is denoted by $\tau(A)$.*

From now on we will assume that $2 \leq m \leq n$. It follows that $1 \leq \tau(A) \leq m$ for all nonzero $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.

Let $\Xi_k^{(r,s)}$ denote the set of all matrices in $\mathbb{M}_{r,s}(\mathbb{S})$ whose term rank is k.

Definition 2.7. *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. If f is a function defined on $\mathbb{M}_{m,n}(\mathbb{S})$ and on $\mathbb{M}_{p,q}(\mathbb{S})$, then T preserves the function f if $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$. If $\mathbb{X}$ is a subset of $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{Y}$ is a subset of $\mathbb{M}_{p,q}(\mathbb{S})$, then T preserves the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. Further, T strongly preserves the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ if and only if $T(A) \in \mathbb{Y}$. Further, we say that T* (strongly) preserves term rank *k if T (strongly) preserves the pair $(\Xi_k^{(m,n)}, \Xi_k^{(p,q)})$.*

Song and Beasley characterized linear transformation on an antinegative semirings without zero-divisor, but in this paper, we characterize linear transformations that preserve term rank between different general Boolean matrix spaces with zero-divisor.

Definition 2.8. *The matrix $A^{(m,n)}$ denotes a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_k)$, $O^{(m,n)}$ is the $m \times n$ zero matrix, I_n is the $n \times n$ identity matrix, $I_k^{(m,n)} = I_k \oplus O_{m-k,n-k}$, and $J^{(m,n)}$ is the $m \times n$ matrix all of whose entries are 1. Let $E_{i,j}^{(m,n)}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and we call $E_{i,j}^{(m,n)}$ a cell. An $m \times n$ matrix $L^{(m,n)}$ is called a full line matrix if*

$$L^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)} \quad \text{or} \quad L^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$$

for some $i \in \{1, \dots, m\}$ or for some $j \in \{1, \dots, n\}$; $R_i^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)}$ is the ith full row matrix and $C_j^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$ is the jth full column matrix. We will suppress the subscripts or superscripts on these matrices when the orders are evident from the context and we write A, O, I, I_k, J, $E_{i,j}$, L, R_i and C_j respectively.

The following is obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.9. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{B}_k)$, we have*

$$\tau(A+B) \leq \tau(A) + \tau(B)$$

and

$$\tau(A) \leq \tau(A+B).$$

Proof. *Let $\tau(A+B) = k$. Then minimal number of lines that contains nonzero entries of $A+B$ is k. Let $\tau(A) = l$. Since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is an antinegative, there isn't an inverse for addition without zero. So $k - l \leq \tau(B)$. And if there exist an entry of B that located out of any A's minimal cover lines, $\tau(A) < \tau(A+B)$. If not, then $\tau(A) = \tau(A+B)$.* ■

Definition 2.10. *If A and B are matrices in $\mathbb{M}_{m,n}(\mathbb{S})$, we say that B dominates A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}(\mathbb{S})$.*

The following is also obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.11. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{B}_k)$, $A \sqsubseteq B$ implies that*

$$\tau(A) \leq \tau(B).$$

Proof. *It is clear by definition of dominating.* ■

Definition 2.12. *As usual, for any matrix A and lists L_1 and L_2 of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows L_1 and columns L_2 from A and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows L_1 and columns L_2 from A.*

Definition 2.13. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $A \circ B$ denotes the Hadamard or Schur product. That is, the $(i,j)^{th}$ entry of $A \circ B$ is $a_{i,j}b_{i,j}$.*

Definition 2.14. *If $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \rightarrow \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation, then T is a (P, Q)*-block-transformation *if there are permutation matrices $P \in \mathbb{M}_p(\mathbb{B}_k)$ and $Q \in \mathbb{M}_q(\mathbb{B}_k)$ such that*

- *$m \leq p$ and $n \leq q$, and $T(A) = P[A \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ or*
- *$m \leq q$ and $n \leq p$, and $T(A) = P[A^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.*

Definition 2.15. *If $\mathbb{S}$ is a commutative antinegative semiring, $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$, then T is a (P, Q, B)*-block-transformation *if there are permutation matrices $P \in \mathbb{M}_p(\mathbb{S})$ and $Q \in \mathbb{M}_q(\mathbb{S})$, and $B \in \mathbb{M}_{m,n}(\mathbb{S})$ which has not zero element such that*

- *$m \leq p$ and $n \leq q$, and $T(A) = P[(A \circ B) \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$ or*
- *$m \leq q$ and $n \leq p$, and $T(A) = P[(A \circ B)^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$.*

3 Term rank preservers of General Boolean matrices

In this section, we obtain some results on the term-rank preservers of general Boolean matrices. All most of these results were studied on the binary Boolean matrices by Song and Beasley([11]).

Definition 3.1. *For a linear transformation $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$, we say that T*

(1) preserves term rank k if $\tau(T(X)) = k$ whenever $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, or equivalently if T preserves the pair $(\Xi_k^{(m,n)},\Xi_k^{(p,q)})$;

(2) strongly preserves term rank k if $\tau(T(X)) = k$ if and only if $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, or equivalently if T strongly preserves the pair $(\Xi_k^{(m,n)},\Xi_k^{(p,q)})$;

(3) preserves term rank if it preserves term rank k for every $k(\leq m)$.

Lemma 3.2. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation that preserves term rank k and term rank 1, then T strongly preserves term rank 1.*

Proof. First to show when $k = 2$, and next $k \geq 3$.

Case 1. Let $\tau(T(A)) = 1$. If $k = 2$ and $\tau(A) \geq 2$, then $A = B + C + D$ with $\tau(B) = 1$, $\tau(C) = 1$ and $\tau(D) \geq 1$ with $\tau(B + C) = 2$. Since $B + C \sqsubseteq A$, $T(B + C) \sqsubseteq T(A)$ and $2 = \tau(T(B) + T(C)) \leq \tau(T(A)) = 1$. This is a contradiction. So T strongly preserves term rank 1.

Case 2. Assume that $k \geq 3$. Suppose a term rank 2 matrix is mapped to a term rank 1 matrix. Without loss of generality, $\tau(T(E_{1,1} + E_{2,2})) = 1$. But then, since T preserves term rank 1, $\tau(T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2}) + T(E_{3,3}) + \cdots + T(E_{k,k})) \leq \tau(T(E_{1,1} + E_{2,2})) + \tau(T(E_{3,3})) + \cdots + \tau(T(E_{k,k}))) = 1 + (k - 2) < k$, a contradiction. Thus, T strongly preserves term rank 1. ∎

Lemma 3.3. *Let $2 \le k \le m \le n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation that preserves term rank k. If T does not preserve term rank 1, then there is some term rank 1 matrix whose image has term rank at least 2.*

Proof. Suppose that T does not preserve term rank 1 and $\tau(T(A)) \le 1$ if $\tau(A) = 1$. Then, there is some cell $E_{i,j}$ such that $a_{i,j}T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since $\tau(E_{1,1} + E_{2,2} + \cdots + E_{k,k}) = k$ and T preserves term rank k, we have $\tau(T(E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k,k})) = k$. Let $X = T(E_{2,2} + \cdots + E_{k,k})$ then we can choose a set of cells $Y = \{F_1, F_2, \cdots, F_k\}$ such that $X \sqsupseteq F_i$ for all $i = 1, \cdots, k$, and $\tau(F_1 + F_2 + \cdots + F_k) = k$. Since $T(E_{2,2} + \cdots + E_{k,k}) = X$, there is some cell in $\{E_{2,2}, \cdots, E_{k,k}\}$ whose image under T dominates two cells in Y, a contradiction. This contradiction establishes the lemma. ∎

Lemma 3.4. *Let $1 \le k \le m \le n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation that preserves term rank k. If $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\tau(A) \le k$ then $\tau(T(A)) \le k$.*

Proof. If $\tau(A) = k$, then $\tau(T(A)) = k$ since T preserves term rank k. Suppose that $\tau(A) = h < k$, and $\tau(T(A)) > k$. Then there exist a matrix $B \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $\tau(A + B) = k$ and hence $\tau(T(A + B)) = k$, but by Lemma 2.9, $\tau(T(A+B)) = \tau(T(A)+T(B)) \ge \tau(T(A)) > k$, a contradiction. Thus $\tau(T(A)) \le k$. ∎

Lemma 3.5. *Let $2 \le k \le m \le n$ and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation that preserves term rank k. If T does not preserve term rank 1, then $\tau(T(J)) \le k + 2$.*

Proof. By Lemma 3.3, if T does not preserve term rank 1, then there is some rank 1 matrix whose image has term rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \ge E_{1,1} + E_{2,2}$.

Suppose that $\tau(T(J)) \ge k + 3$. Then, $\tau(T(J)[3, \cdots, p|3, \cdots, q]) \ge k - 1$. Without loss of generality, we may assume that $T(J)[3, \cdots, p|3, \cdots, q] \sqsupseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Thus, there are $k - 1$ cells, $F_3, F_4, \cdots, F_{k+1}$ such that $T(F_3 + F_4 + \cdots + F_{k+1}) \sqsupseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Then, $T(E_{1,1} + E_{1,2} +$

$F_3 + F_4 + \cdots + F_{k+1}) \sqsupseteq I_{k+1}$. But, $\tau(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1}) \leq k$ while $\tau(T(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1})) \geq k+1$, a contradiction. Thus, $\tau(T(J)) \leq k+2$. ■

Lemma 3.6. *Let $1 \leq k < r, s$ and $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$. If $\tau(E_{1,1}+\cdots+E_{k,k}+A) \geq k+1$ and $A[k+1, \cdots, r|k+1, \cdots, s] = O$, then there is some $i, 1 \leq i \leq k$, such that $\tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1} + \cdots + E_{k,k} + A) \geq k+1$.*

Proof. Suppose that $B = E_{1,1} + \cdots + E_{k,k} + A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\tau(B) \geq k+1$. Then there are $k+1$ cells $F_1, F_2, \cdots, F_{k+1}$ such that $B \sqsupseteq F_1 + F_2 + \cdots + F_{k+1}$ and $\tau(F_1 + F_2 + \cdots + F_{k+1}) = k+1$. If $F_1 + F_2 + \cdots + F_{k+1} \sqsupseteq I_k \oplus O$ then one cell F_j must be a cell $E_{a,b}$ where $a, b \geq k+1$, which contradicts the assumption $A[k+1, \cdots, r|k+1, \cdots, s] = O$. Thus $F_1 + F_2 + \cdots + F_{k+1}$ does not dominate $I_k \oplus O$. That is, there is some $i, 1 \leq i \leq k$, such that $\tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1} + \cdots + E_{k,k} + A) \geq k+1$. ■

Lemma 3.7. *Let $1 \leq k < l \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation that preserves term rank k and term rank l, then T preserves term rank 1.*

Proof. We prove this lemma by 3 cases according to distance between k and l.

Case 1. Let $k+3 \leq l$. Suppose that T does not preserve term rank 1. By Lemma 3.3, there is some term rank 1 matrix whose image has term rank at least 2. Let A be such a term rank 1 matrix. Then, A is dominated by a row or column and the image of the sum of two cells in that line has term rank at least two. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2}) \sqsupseteq E_{1,1}+E_{2,2}$. Now, by Lemma 3.5, if $B = T(C)$ is in the image of T, $\tau(B) \leq k+2 < l$. But if we take $B = T(I_l)$, then $T(I_l)$ must have term rank l, a contradiction.

That is, $\tau(T(A)) \leq 1$. Since A was an arbitrary term rank 1 matrix, T preserves term rank 1.

Case 2. Let $k+1 = l$. If $k = 1$, the lemma vacuously holds. Suppose that $k \geq 2$.

Suppose that T does not preserve term rank 1. Then there is some matrix of term rank 1 whose image has term rank at least 2. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \sqsupseteq E_{1,1} + E_{2,2}$. By Lemma 3.5, we have that $\tau(T(J)) \le k+2$. Since T preserves term rank $k+1$, $\tau(T(J)) \ge k+1$.

Thus, $\tau(T(J)) = k+i$ for either $i = 1$ or $i = 2$. Now, we may assume that for some r, s with $r + s = k+i$, $T(J)[r+1, \cdots, p|s+1, \cdots, q] = O$. Further, we may assume, without loss of generality, that there are $k+i$ cells $F_1, F_2, \cdots, F_{k+i}$ such that $T(F_l) \sqsupseteq E_{l,k+i-l+1}$ for $l = 1, \cdots, k+i$. Suppose the image of one of the cells in $F_1, F_2, \cdots, F_{k+i}$ dominates more than one cell in $\{E_{1,k+i}, E_{2,k+i-1}, \cdots, E_{k+1,i}\}$. Say, without loss of generality, that $T(F_1) \sqsupseteq E_{1,k+i} + E_{2,k+i-1}$, then, $T(F_1 + F_3 + \cdots + F_{k+1}) \sqsupseteq E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}$, a contradiction since $\tau(F_1 + F_3 + \cdots + F_{k+1}) \le k$, and hence $\tau(T(F_1 + F_3 + \cdots + F_{k+1})) \le k$, and $\tau(E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}) = k+1$. It follows that for each $j = 1, \cdots, k+1$, if $T(F_l) \sqsupseteq E_{j,k+i-j+1}$ then $l = j$ since $T(F_j) \sqsupseteq E_{j,k+i-j+1}$ is unique. Further, by permuting we may assume that $F_1 + F_2 + \cdots + F_k \sqsubseteq \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix}$.

Now, let $O \ne A \in \mathbb{M}_{m,n}(\mathbb{B})$ have term rank 1, and suppose that $A[1, 2, \cdots, k|1, 2, \cdots, n] = O$ and $A[K = 1, \cdots m|1, \cdots, k] = O$. So that $A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_1 \end{bmatrix}$. If $T(A)[k+1, \cdots, p|1, i] = O$, then, since $\tau(F_1 + \cdots + F_k + A) = k+1$, $\tau(T(F_1 + \cdots + F_k + A)) = k+1$. Applying Lemma 3.6, we have that there is some j such that $\tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k+1$. But $\tau(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A) = k$ while $\tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k+1$, a contradiction. So we must have that $T(E_{k+1,1})[k+1, \cdots, p|1, i] \ne O$. If $T(E_{k+1,1})[k+1, \cdots, p|1, i] \ne O$ then $\tau(T(F_1 + \cdots + F_k + E_{k+1,1})) = k+1$, a contradiction since $\tau(F_1 + \cdots + F_k + E_{k+1,1} = k$. Suppose that the $(k, i+1)$ entry of $T(E_{k,k+1})$ is nonzero, then, $\tau(T(F_1 + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+1})) = k+1$, a contradiction, since $\tau(F_1 + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+1}) = k$.

Consider $T(F_1 + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2})$. This must have term rank $k+1$ and dominates $E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k-1,i+2} + E_{k+1,j}$ for some $j \in \{1, i\}$. Thus, by Lemma 3.6, there is some cell in $\{F_1, \cdots, F_{k-1}\}$, say F_j such that $\tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2})) = k+1$. But $\tau(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2}) = k$, a contradiction.

It follows that T must preserve term rank 1.

Case 3. Let $k+2 = l$ and $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.

Subcase 3-1. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) \geq k+2$. Let $A_1, A_2, \cdots,$ A_{k+1} be matrices of term rank 1 such that $A = A_1 + A_2 + \cdots + A_{k+1}$. Without loss of generality we may assume that $T(A) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{k+2,k+2}$ and, since the image of some A_i must have term rank at least 2, we may assume that $\tau(T(A_1 + A_2 + \cdots + A_i)) \geq i + 1$, for every $i = 1, 2, \cdots k + 1$. But then $\tau(A_1 + A_2 + \cdots + A_k) = k$ while $\tau(T(A_1 + A_2 + \cdots + A_k)) \geq k+1$, a contradiction, Thus if $\tau(A) = k + 1$, $\tau(T(A)) \leq k + 1$.

Subcase 3-2. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) = s \leq k$. Without loss of generality, we may assume that $A = E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1}$ and $T(A) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Then there are s members of $\{T(E_{1,1}), T(E_{2,2}), \cdots, T(E_{k+1,k+1})\}$ whose sum dominates $E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Say, without loss of generality, that $T(E_{1,1} + E_{2,2} + \cdots + E_{s,s}) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Now, $\tau(A + E_{k+2,k+2}) = k+2$ so that $\tau(T(A + E_{k+2,k+2})) = k + 2$. But since $\tau(T(A + E_{k+2,k+2})) = \tau((T(A) + T(E_{k+2,k+2})) \leq \tau(T(A)) + \tau(T(E_{k+2,k+2}))$, it follows that $\tau(T(E_{k+2,k+2})) \geq k + 2 - s$ and there are s members of $\{T(E_{1,1}), T(E_{2,2}), \cdots, T(E_{k+1,k+1})\}$ whose sum together with $T(E_{k+2,k+2})$ has term rank $k+2$, say $\tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2$. Since $s \leq k$, $\tau(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2}) \leq k + 1$ and $\tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2$. By Case 1, we again arrive at a contradiction.

Therefore T strongly preserves term rank $k + 1$. And by Case 2, T preserves term rank 1.

Thus we prove completely the lemma by 3 cases. ■

Lemma 3.8. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation that strongly preserves term rank k, Then T preserves term rank $k - 1$.*

Proof. If $k = 2$, the lemma holds. Suppose that $k \geq 3$.

Let $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and $\tau(A) = k - 1$, and suppose that $\tau(T(A)) = s < k - 1$. Without loss of generality, we may assume that $\tau(T(E_{1,1} + \cdots + E_{k-1,k-1})) = s < k-1$. Since $\tau(T(E_{1,1} + \cdots + E_{k,k})) = k$, we have that $\tau(T(E_{k,k})) \geq k - s$. Without loss of generality we may assume that $T(E_{1,1} + \cdots + E_{k,k}) \sqsupseteq E_{1,1} + \cdots + E_{k,k}$ and that $T(E_{k,k}) \sqsupseteq E_{t+1,t+1} + \cdots + E_{k,k}$ for some $t \leq s$. Then, there are t cells $\{E_{i_1,i_1}, \cdots, E_{i_t,i_t}\}$ in $\{E_{1,1}, \cdots, E_{k,k}\}$ such that $T(E_{i_1,i_1} + \cdots + E_{i_t,i_t}) \sqsupseteq$

$E_{1,1} + \cdots + E_{t,t}$. Then $T(E_{i_1,i_1} + \cdots + E_{i_t,i_t} + E_{k,k}) \sqsupseteq E_{1,1} + \cdots + E_{k,k}$. Thus $\tau(T(E_{i_1,i_1}+\cdots+E_{i_t,i_t}+E_{k,k})) = k$. But $\tau(E_{1,1}+\cdots+E_{t,t}+E_{k,k}) = t+1 \leq s+1 < (k-1)+1 = k$, which contradicts the assumption of T. Hence $\tau(T(A)) \geq k-1$. Further, $\tau(T(A)) \leq k-1$, since T strongly preserves term rank k. Thus, T preserves term rank $k-1$. ∎

Lemma 3.9. *Let $2 \leq k < m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation that strongly preserves term rank k, then T preserves term rank 1.*

Proof. By Lemma 3.8, T preserves term rank $k-1$. By Lemma 3.7, T preserves term rank 1. ∎

Now we provide characterizations of linear transformations $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ that preserve term ranks k and l, where $1 \leq k < l \leq m \leq n$.

Theorem 3.10. *Let $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$. Then T strongly preserves term rank 1 if and only if T is a (P, Q)-block-transformation. (Necessarily, either $m \leq p$ and $n \leq q$, or $m \leq q$ and $n \leq p$.)*

Proof. It is routine to show that if T is a (P, Q)-block transformation, then T strongly preserves term rank 1.

Assume that T strongly preserves term rank 1. Then, the image of each line in $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a line in $\mathbb{M}_{p,q}(\mathbb{B}_k)$. We may assume that either $T(R_1^{(m,n)}) \leq R_1^{(p,q)}$ or $T(R_1^{(m,n)}) \leq C_1^{(p,q)}$.

Case 1. $T(R_1^{(m,n)}) \leq R_1^{(p,q)}$. Suppose that $T(C_j^{(m,n)}) \leq R_i^{(p,q)}$. Then, since $E_{1,j}^{(m,n)}$ is in both $R_1^{(m,n)}$ and $C_j^{(m,n)}$ and since $T(E_{1,j}^{(m,n)}) \neq O$, we must have $i = 1$. But then, for $j \neq k$ $T(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) \leq R_1^{(m,n)}$ and hence, has term rank 1. But $\tau(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) = 2$, a contradiction. Thus the image of any column is dominated by a column. Similarly, the image of any row is dominated by a row. Further, since the sum of two rows (columns) has term rank 2, the image of distinct rows (columns) must be dominated by distinct rows (columns). Let $\phi : \{1, \cdots m\} \to \{1, \cdots, p\}$ be a mapping defined by $\phi(i) = j$ if $T(R_i^{(m,n)}) \leq R_j^{(p,q)}$ and define $\theta : \{1, \cdots n\} \to \{1, \cdots, q\}$ by $\theta(i) = j$ if $T(C_i^{(m,n)}) \leq C_j^{(p,q)}$. Then, it is easily seen that ϕ and θ are one-to-one mappings, and hence, $m \leq p$ and $n \leq q$.

Let $\phi' : \{1, \cdots, p\} \to \{1, \cdots, p\}$ and $\theta' : \{1, \cdots, q\} \to \{1, \cdots, q\}$ be one-to-one mappings such that $\phi' \mid_{\{1, \cdots m\}} = \phi$ and $\theta' \mid_{\{1, \cdots n\}} = \theta$. Let $P_{\phi'}$ and $Q_{\theta'}$ denote the permutation matrices corresponding to the permutations ϕ' and θ'.

In this case we have that $m \leq p$ and $n \leq q$, and $T(A) = P_{\phi'}[A \oplus O]Q_{\theta'}$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, that is T is a (P, Q)-block-transformation.

Case 2. $T(R_1^{(m,n)}) \leq C_1^{(p,q)}$. As in case 1, a parallel argument shows that $m \leq q$ and $n \leq p$, and $T(A) = P[A^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, and consequently that T is a (P, Q)-block-transformation. ■

Corollary 3.11. *Let $1 < k \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation. Then T preserves term rank 1 and term rank k if and only if T is a (P, Q)-block-transformation.*

Proof. By Lemma 3.2, T strongly preserves term rank 1. By Theorem 3.10, the corollary follows. ■

Theorem 3.12. *Let $1 \leq k < l \leq m \leq n$ and $k + 1 < m$. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation that preserves term rank k and term rank l, or if T strongly preserves term rank k, then T is a (P, Q)-block-transformation.*

Proof. By hypothesis, Lemma 3.7 or Lemma 3.9, T preserves term rank 1. By Corollary 3.11, the theorem follows. ■

Theorem 3.13. *Let $1 \leq k < l \leq m \leq n$ and $k + 1 < m$. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ is a linear transformation that strongly preserves term rank k, then T is a (P, Q)-block-transformation.*

Proof. By Lemma 3.2, T strongly preserves term rank 1. By Theorem 3.10, the theorem follows. ■

4 Equivalent conditions of term rank preservers on the general Boolean matrices

Throughout this section, we characterize term rank preservers over different general Boolean matrix spaces.

At first, provide two examples.

Example 4.1. *Let $T(A) = P[(A \circ B) \oplus O]Q$, and*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \{a\} & \{b\} \\ \{a\} & \{a, b\} \end{pmatrix}.$$

Then for $A = \begin{pmatrix} \{b\} & \{b\} \\ \{b\} & \{b\} \end{pmatrix}$, $T(A) = \begin{pmatrix} 0 & 0 & \{b\} \\ 0 & 0 & \{b\} \end{pmatrix}$. *So* $\tau(A) = 2$, *but* $\tau(T(A)) = 1$.

Thus, in general Boolean algebra, (P, Q, B)-block-transformation does not preserve term rank.

Example 4.2. *Let $T(A) = P[(A \circ B) \oplus O]Q$, and*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = J.$$

Then for $A = \begin{pmatrix} \{a\} & \{a\} \\ \{b\} & \{b\} \end{pmatrix}$, $T(A) = \begin{pmatrix} \{b\} & 0 & \{b\} \\ \{a\} & 0 & \{a\} \end{pmatrix}$. *Thus* $\tau(A) = 2$, *and* $\tau(T(A)) = 2$. *Actually,* $(A \circ J)$ *is a* A.

That is , for any general Boolean algebra, (P, Q)-block transformation preserve term rank, which is a special case of (P, Q, B)-block-transformation with unit b_{ij}.

We provide characterizations of linear transformations $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ that preserve term rank.

Definition 4.3. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ *and define* $\overline{A} \in \mathbb{M}_{m,n}(\mathbb{B})$ *to be the matrix* $[\overline{a}_{i,j}]$ *where* $\overline{a}_{i,j} = 1$ *if and only if* $a_{i,j} \neq 0$. $\overline{A}$ *is called the* support *or* pattern *of* A. *Clearly* $\tau(\overline{A}) = \tau(A)$. *Let* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ *be a linear transformation. Define* $\overline{T} : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ *by* $\overline{T}(E_{i,j}) = \overline{T(E_{i,j})}$, *and extend linearly. Then* $\overline{T} : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ *is a linear transformation over binary Boolean semiring.*

Lemma 4.4. *Let* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ *be a linear transformation. Then* T *preserves term rank* k *if and only if* $\overline{T}$ *preserves term rank* k, *for any* $1 \leq k \leq m$.

Proof. Let $\overline{A} \in \mathbb{M}_{m,n}(\mathbb{B})$ with $\tau(\overline{A}) = k$. There exist $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ with $\tau(A) = k$. Then $\tau(T(A)) = k$. Since $\overline{T}(\overline{A})$ has the same zero pattern as T(A), $\tau(\overline{T}(\overline{A})) = k$. Conversely, let $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ with $\tau(A) = k$. There exist $\overline{A} \in \mathbb{M}_{m,n}(\mathbb{B})$ with $\tau(\overline{A}) = k$. Then $\tau(\overline{T}(\overline{A})) = k$. Since T(A) has the same zero pattern as $\overline{T}(\overline{A})$, $\tau(T(A)) = k$. ∎

Theorem 4.5. *Let* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{p,q}(\mathbb{B}_k)$ *be a linear transformation. Then the following are equivalent*:

1. T *preserves term rank*;
2. T *preserves term rank* k *and term rank* l, *with* $1 \leq k < l \leq m \leq n$ *and* $k + 1 < m$;
3. T *strongly preserves term rank* h, *with* $1 \leq h < m \leq n$;
4. T *is a* (P, Q)*-block transformation.*

Proof. 1 implies 2 and 3 by definition of preserving term rank. Let A be any matrix in $\mathbb{M}_{m,n}(\mathbb{B})$ with $\tau(A) = k$. $\tau(T(A)) = \tau(P[(A \oplus O)]Q = \tau(A \oplus O) = \tau(A)$. Thus 4 implies 1, 2 and 3. In order to show that 2(or 3) implies 4, assume that T preserves term rank k and term rank l, with $1 \leq k < l \leq m \leq n$. By Lemma 4.4, $\overline{T}$ preserves term rank k and term rank l, with $1 \leq k < l \leq m \leq n$.. Thus, by Theorem 3.12, $\overline{T}$ is a (P, Q)-block transformation. Thus, T is a (P, Q, B)-block transformation. But the entries of B must be unit. Thus $B = J$ and T is a (P, Q)-block transformation.

In order to show that 3 implies 4, if we apply Lemma 4.4 and Theorem 3.13, the proof is parallel to the above.

■

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<국문초록>

서로 다른 일반적인 부울 行列 空間 사이의 項 係數 保存者

본 논문에서는 일반적인 $m\times n$ 부울 行列 空間에서 일반적인 $p\times q$ 부울 行列 空間으로의 線型 變換을 연구하였다. 서로 다른 일반적인 부울 行列 空間들 사이의 項 係數를 保存하는 線型 變換을 특성화하였다. 이 결과는 $m\times n$인 二項 부울 行列 空間상의 項 係數를 保存하는 線型 演算子에 대한 연구 결과를 확장한 것이다.

行列 A의 項 係數는 行列 A의 0 아닌 성분들을 모두 포함하는 h개의 行들과 $k-h$개의 列들에 대해 最小의 k를 의미한다. A의 項 係數는 $\tau(A)$로 표시한다.

$\Xi_k^{(r,s)}$는 $r\times s$ 부울 行列들의 집합 $\mathbb{M}_{(r,s)}(\mathbb{B}_k)$상의 項 係數가 k인 모든 부울 行列들의 집합이라고 하자.

$T: \mathbb{M}_{m,n}(\mathbb{B}_k)\rightarrow\mathbb{M}_{p,q}(\mathbb{B}_k)$를 $m\times n$ 부울 行列 空間에서 일반적인 $p\times q$ 부울 行列 空間으로의 線型 變換이라고 하자. 만일 f가 行列 空間들에서 정의된 함수일 때, T가 行列 空間의 모든 行列 A에 대하여 $f(T(A))=f(A)$를 만족하면 T는 함수 f를 保存한다고 한다. 그리고 行列 空間의 부분집합 $\mathbb{X}$와 行列 空間의 부분집합 $\mathbb{Y}$에 대하여, $A\in\mathbb{X}$이면 $T(A)\in\mathbb{Y}$를 만족할 때 T는 順序雙 $(\mathbb{X},\mathbb{Y})$를 保存한다고 한다. 더욱이 T가 順序雙 $(\Xi_k^{(m,n)},\Xi_k^{(p,q)})$를 (강하게) 保存하면, T는 項 係數 k를 (강하게) 保存한다고 한다.

宋錫準 교수와 Beasley 교수는 零因子를 포함하지 않는 非陰의 半環들 상에서 項 係數를 保存하는 線型 變換을 특성화하였다. 그러나 이 논문에서는 零因子를 포함하는 서로 다른 일반적인 부울 行列 空間상에서의 項 係數를 保存하는 線型 變換을 특성화하였다. 이 논문의 주요 연구 결과는 다음과 같다.

定理. Let $T: \mathbb{M}_{m,n}(\mathbb{B}_k)\rightarrow\mathbb{M}_{p,q}(\mathbb{B}_k)$ be a linear transformation. Then the following are equivalent:

1. T preserves term rank;
2. T preserves term rank k and term rank l, with $1\leqq k<l\leqq m\leqq n$ and $k+1<m$;
3. T strongly preserves term rank h, with $1\leqq h<m\leqq n$;
4. T has the form $T(X)=P(X\oplus O)Q$, where P, Q are permutation matrices of order p and q, respect.

<감사의 글>

처음 대학원을 접수하고자 할 때만 해도, 대학 시절 그리 열심히 공부에 전념 하지 못했던 제가 과연 대학원 생활을 잘 할 수 있을까 하는 걱정과 제 또래가 아닌 어린 학우들과 같이 공부하는 것에 대해 부담을 느끼고 있었습니다. 하지만 대학원을 들어와 공부하고자 했던 게 자의에서 나온 것이라서 해 볼만 한 도전이라고도 생각을 했습니다. 그리고 2년이 지난 지금에 와서 드는 생각은 대학원에 들어오길 잘했구나 하는 만족감입니다.

이전에 대학 생활을 하는 동안, 그리고 임용고시를 준비하는 동안에 공부를 하면서 대수계열의 과목에 대한 자신감이 없었습니다. 대학원을 진학하면서 스스로 다짐하기를 제가 자신없어하는 대수 분야를 어떻게든 이해하고 싶었기에 전공으로 대수를 선택하였습니다. 전공을 선택하기 전 주위 분들에게 물어보고 상담도 하면서 대학원을 진학하는데 자신 있는 분야를 선택해야지 배우고자 하는데 의의를 두면 되겠냐는 말씀을 많이 들었습니다. 그렇지만 하고 싶은 것이 확실하다는 생각을 하게 되어 결국에 자신이 없는 분야였던 대수 분야를 선택하였고, 지금은 그 선택에 대해 전혀 후회가 없습니다.

논문을 쓸 준비를 하면서 많은 고민을 했습니다. '과연 제대로 논문을 끝마칠 수 있을까?', '논문은 깔끔하게 정리되어 나올까?' 같은 고민들을 하였습니다. 그러나 한편으로는 욕심도 없지 않았습니다. 어떻게든 써야 할 논문 이왕이면 멋지게 써내고픈 그런 욕심이었습니다. 정말이지 이제 막 덧셈 배운 어린 아이가 내일 당장 학교에서 학생들을 가르치는 선생님이 되겠다는 것과 다를 바 없는 생각이었습니다. 결과물을 낸 지금 드는 생각은 아직 배울게 많은 제가 이 만큼의 결과를 낸 것으로도 충분히 만족스럽다는 것입니다.

현재 제가 하고 싶은 공부가 끝이 난 상태가 아니기에, 그리고 대학원 석사 과정을 통해 기초 공사를 조금은 다져진 상태라고 생각하기에 앞으로 더욱 공부하는데 흥미를 잃지 않고 전진해 나갈 생각입니다.

대학원에 들어와 수업을 통해 도움을 주셨던 양영오 교수님 · 송석준 교수님 · 방은숙 교수님 · 윤용식 교수님 · 정승달 교수님 · 진현성 교수님들께 감사의 마음을 전합니다. 그리고 수업을 통한 가르침은 받지 않았으나, 공부하는 중간에 상담을 통해 세상을 보는 눈을 넓게 해주신 유상욱 교수님 · 강경태 선생님 · 조교 선생님들과 수학교육과 이경언 교수님께도 감사의 마음을 전합니다.
같이 공부하면서 많은 나이 차 때문에 대하기 힘들었을 310호 동기들과 학우들에게도 감사하다는 말을 전합니다.

마지막으로 나이 먹고 공부를 다시 시작한 저를 곁에서 응원해준 저의 사랑하는 어머님, 남동생 진석이, 여동생 예진이에게 고맙고, 감사한다는 말을 전하고, 이렇게 공부를 다시 시작할 수 있게 동기를 부여해준 우리 나영이에게 고맙고, 사랑한다는 말을 전합니다.

2013년 12월