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A study on the effect

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A study on the effect

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CONTENTS

< **Abstract**>

A study on the effect

We introduce the concepts of effects (or called fuzzy events) and observable (or called fuzzy random variable) as the generalizations of event and random variable, respectively. Also, we introduce the concept of sigma-morphism and we prove some basic properties and continuity of sigma-morphism as a probability measure on the set of effects. There are various types of mean value and variance for fuzzy sets. We study mean value and variance defined by Christer Carlsson and Robert Fullér. For two independent random variables A and B, the expectation of AB equals the product of two expectations of A and B. We investigate the corresponding property for independence which is one-sided fuzzy set. We show that the mean value of product AB of two one-sided fuzzy sets A and B equals the product of two mean values of one-sided fuzzy sets A and B. And, we calculate the possibilistic mean value, variance and covariance of one-sided fuzzy sets and their products.

1. Introduction

Since L. A. Zadeh introduced the concept of fuzzy sets, the theory of fuzzy sets has been developed in various aspects. In some senses, many theories in fuzzy mathematics can be considered as a generalization of the original theory. In probability theory, many researchers have tried to generalize the concepts of events and random variables in probability theory.

In probability theory, the imprecision comes from our incomplete knowledge of the system but the random variables(measurements) still have precise values. But, in fuzzy theory, we also have an imprecision in our measurements, and so random variables must be replaced by fuzzy random variables and events by fuzzy events. In this sense, S. Gudder introduced the concepts of effects(fuzzy events), observable(fuzzy random variable) and their distribution. Also, he introduced the concept of σ-morphism on the set of effects.

In section 2, we introduced the basic concepts of fuzzy sets. And we introduced the concept of fuzzy numbers and the operations(addition, subtraction, multiplication and division) for them. Especially, we introduced the concepts of triangular fuzzy numbers, trapezoidal fuzzy numbers and quadratic fuzzy numbers and the results of four operations for these fuzzy numbers.

In section 3, we introduced the concept of effect, observable and σ-morphism as a probability measure on the set of effects and some basic properties of σ-morphism. And we proved some properties about σ-morphism. The main theorem in section 3 is the continuity of σ-morphism that can be considered as a generalization of the continuity of probability measure.

In section 4, we introduced the concepts of possibilistic mean value and variance of fuzzy numbers defined by C. Carlsson and R. Fullér [7] and calculate the mean value and variance of some special fuzzy numbers introduced in section 2. To develop our calculations, we define the concepts of one-sided fuzzy set. The main result in section 4 is that, in some special case, the mean of the product of two fuzzy sets is the product of means of each fuzzy sets. This result can be considered as the similar result which is well-known in the independence of events in probability theory.

2. Preliminaries

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0,1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0,1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A *fuzzy set* A on X is a function from X to the interval $[0,1]$. The function is called the *membership function* of A.

Let A be a fuzzy set on X with a membership function μ_A. Then A is a subset of X that has no sharp boundary. A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

Elements with a zero degree of membership are normally not listed.

If X is a finite set $\{x_1, \cdots, x_n\}$, a fuzzy set A on X is expressed as

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i.$$

If X is not finite, we write

$$A = \int_X \mu_A(x)/x.$$

Two fuzzy sets A and B are said to be equal, denoted by $A = B$, if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$.

Definition 2.2. A γ*-level set* of a fuzzy set A on $\mathbb{R}$ is defined by $[A]^\gamma = \{x \in \mathbb{R} | \mu_A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = cl\{x \in \mathbb{R} | \mu_A(x) > \gamma\}$ if $\gamma = 0$.

Definition 2.3. A fuzzy set A on $\mathbb{R}$ is *convex* if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

for all x_1, x_2 in $\mathbb{R}$ and λ in $[0,1]$.

Definition 2.4. A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if
(1) there exists exactly one $x_0 \in \mathbb{R}$ such that $\mu_A(x_0) = 1$,
(2) $\mu_A(x)$ is piecewise continuous.

Definition 2.5. A *triangular fuzzy number* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a-\alpha, a+\beta \leq x, \\ (x-a+\alpha)/\alpha & \text{if} \quad a-\alpha \leq x < a, \\ (a+\beta-x)/\beta & \text{if} \quad a \leq x < a+\beta. \end{cases}$$

The above triangular fuzzy set is denoted by $A = (\alpha, a, \beta)$.

Definition 2.6. A *quadratic fuzzy number* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < \alpha, \beta \leq x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2+1 & \text{if} \quad \alpha \leq x < \beta, \end{cases}$$

where $a > 0$. The above quadratic fuzzy set is denoted by $A = [\alpha, k, \beta]$.

Definition 2.7. A fuzzy set A on $\mathbb{R}$ having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_4 \le x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ 1, & a_2 \le x < a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x < a_4. \end{cases}$$

is called a *trapezoidal fuzzy set*. The above trapezoidal fuzzy set is denoted by $A = (a_1, a_2, a_3, a_4)$.

The addition, multiplication and scalar multiplication of fuzzy sets are defined by the extension principle [5].

Definition 2.8. For two fuzzy sets A and B on $\mathbb{R}$, the *addition, subtraction, multiplication* and *division* are defined as

(1) addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R},$$

(2) subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R},$$

(3) multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R},$$

(4) division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R}.$$

Theorem 2.9. ([5]) For two triangular fuzzy numbers $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, we have

(1) $A(+)B = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3)$,

(2) $A(-)B = (a_1 - b_3,\ a_2 - b_2,\ a_3 - b_1)$,

(3) $A(\cdot)B$ and $A(/)B$ need not to be triangular fuzzy numbers.

Example 2.10. Let $A = (2, 5, 7)$ and $B = (1, 4, 6)$ be the triangular fuzzy numbers. Then we have

(1) $A(+)B = (3, 9, 13)$,

(2) $A(-)B = (-4, 1, 6)$,

(3) $A(\cdot)B$ and $A(/)B$ need not to be triangular fuzzy numbers.

Theorem 2.11. ([5]) For two quadratic fuzzy numbers $A = [x_1, k, x_2]$ and $B = [x_3, m, x_4]$, we have

(1) $A(+)B = [x_1 + x_3, k + m, x_2 + x_4]$,

(2) $A(-)B = [x_1 - x_4, k - m, x_2 - x_3]$,

(3) $\mu_{A(\cdot)B}(x) = 0$ on the interval $[x_1x_3, x_2x_4]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = km$. Note that $A(\cdot)B$ needs not to be a quadratic fuzzy number,

(4) $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{x_1}{x_4}, \frac{x_2}{x_3}]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{k}{m}$. Note that $A(/)B$ needs not to be a quadratic fuzzy number.

Example 2.12. Let $A = [1, 2, 3]$ and $B = [2, 7, 12]$ be the quadratic fuzzy numbers. Then we have

(1) $A(+)B = [3, 9, 15]$,

(2) $A(-)B = [-11, -5, 1]$,

(3) $A(\cdot)B$ and $A(/)B$ need not to be quadratic fuzzy numbers.

Theorem 2.13. ([5]) For two trapezoidal fuzzy sets $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, we have

(1) $A(+)B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4)$,

(2) $A(-)B = (a_1 - b_4,\ a_2 - b_3,\ a_3 - b_2,\ a_4 - b_1)$,

(3) $A(\cdot)B$ and $A(/)B$ need not to be trapezoidal fuzzy sets.

Example 2.14. Let $A = (1, 3, 5, 8)$ and $B = (-1, 2, 5, 9)$ be the trapezoidal fuzzy sets. Then we have

(1) $A(+)B = (0, 5, 10, 17)$,

(2) $A(-)B = (-8, -2, 3, 9)$,

(3) $A(\cdot)B$ and $A(/)B$ need not to be trapezoidal fuzzy sets.

3. σ-morphism on the set of effects

3.1 The basic properties of σ-morphism on the set of effects

The basic structure is a measurable space $(\Omega, \mathcal{F})$ where Ω is a sample space consisting of *outcomes* and $\mathcal{F}$ is a σ-field of *events* in Ω corresponding to some probabilistic experiment. It is useful to identify an event A with its indicator function I_A. If μ is a probability measure on $(\Omega, \mathcal{F})$, then $\mu(A)$ is interpreted as the probability that the event A occurs. A measurable function $f : \Omega \to \mathbb{R}$ is called a *random variable*. The *expectation* of f is defined by $E[f] = \int f d\mu$. Denoting the Borel σ-algebra on the real line $\mathbb{R}$ by $\mathcal{B}(\mathbb{R})$, the *distribution* of f is the probability measure μ_f on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by $\mu_f(B) = \mu(f^{-1}(B))$. We interpret $\mu_f(B)$ as the probability that f has a value in the set B. Notice that $\mu(I_A) = \mu(A)$ for any $A \in \mathcal{A}$ so the identification of A with I_A carries directly over to probabilities. In particular, this identification enables us to give simple proofs of basic properties of probabilities.

The distribution of f can be written

$$\mu_f(B) = \mu(f^{-1}(B)) = \mu(I_{f^{-1}(B)}) = \mu(X_f(B))$$

and we call $\mu_f(B) = \mu(X_f(B))$ the distribution of X_f.

Definition 3.1. A random variable $f : \Omega \to [0, 1]$ is called an *effect* or *fuzzy event*.

Thus, an effect is just a measurable fuzzy subset of Ω. The set of effects is denoted by $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{F})$.

Definition 3.2. If μ is a probability measure on $(\Omega, \mathcal{F})$ and $f \in \mathcal{E}$, we define the *probability* of f to be its expectation $E[f] = \int f d\mu$.

If (f_i) is an increasing sequence in $\mathcal{E}$, then by the monotone convergence theorem, $E[\lim f_i] = \lim E[f_i]$ so E is countably additive. Stated in another way, if a sequence (f_i) in $\mathcal{E}$ satisfies $\sum f_i \in \mathcal{E}$, then $E[\sum f_i] = \sum E[f_i]$.

We call $\mathcal{E}_c(\Omega, \mathcal{F}) = \{I_A : A \in \mathcal{F}\}$ the *set of crisp effects*. Since we are describing probability theory in terms of $\mathcal{E}_c(\Omega, \mathcal{F})$, we would also like to describe random variables in terms of $\mathcal{E}_c(\Omega, \mathcal{F})$. If $f : \Omega \to \mathbb{R}$ is a random variable, define $X_f : \mathcal{B}(\mathbb{R}) \to \mathcal{E}_c(\Omega, \mathcal{F})$ by $X_f(B) = I_{f^{-1}(B)}$. Then X_f satisfies the conditions

$$X_f(\mathbb{R}) = I_{f^{-1}(\mathbb{R})} = 1$$

and if $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, then

$$X_f(\cup A_i) = I_{f^{-1}(\cup A_i)} = I_{\cup f^{-1}(A_i)} = \sum I_{f^{-1}(A_i)} = \sum X_f(A_i).$$

Conversely, if $X_f : B(\mathbb{R}) \to \mathcal{E}_c(\Omega, \mathcal{F})$ satisfies these two conditions, then it can be shown that there exists a unique random variable $f : \Omega \to \mathbb{R}$ such that $X_f = X$. We call X_f the crisp observable corresponding to f.

It is frequently useful to consider more general random variables and crisp observables. Let $(\Lambda, \mathcal{B})$ be another measurable space and let $f : \Omega \to \Lambda$ be a measurable function. we call f a r*andom variable with value space* Λ and the mapping $X_f : \mathcal{B}(\mathbb{R}) \to \mathcal{E}_c(\Omega, \mathcal{F})$ given by $X_f(B) \to I_{f^{-1}(B)}$ is the corresponding *crisp observable with value space* Λ. We now give the general definition of an observable.

Definition 3.3. Let $\mathcal{B}$ be a σ-field of Λ. An *observable* is a map $X : \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{F})$ such that $X(\Lambda) = 1_\Omega$ and if $B_i \in \mathcal{B}(i = 1, 2, 3, \cdots)$ are mutually disjoint, then $X(\cup B_i) = \sum X(B_i)$ where the convergence of the summation is pointwise.

If $X(B)$ is crisp for every $B \in \mathcal{B}$, then X is crisp. If μ is a probability measure on $(\Omega, \mathcal{F})$, then the *distribution* of X is the probability measure μ_X on $(\Lambda, \mathcal{B})$ given by $\mu_X(B) = \mu(X(B))$. Note that μ_X is indeed a probability measure because $\mu_X(\Lambda) = 1$ and if $B_i \in \mathcal{B}$ are mutually disjoint, then by the monotone convergence theorem,

$$\mu_X(\cup B_i) = \mu(X(\cup B_i)) = \mu\left(\sum X(B_i)\right) = \sum \mu(X(B_i)) = \sum \mu_X(B_i)$$

Example 3.4. If $f : (\Lambda, \mathcal{B}) \to (\Omega, \mathcal{F})$ is a measurable function, the corresponding crisp observable $X_f : \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$.

To summarize we can describe probability theory in an equivalent way by replacing events by crisp effects ($A \to I_A$), probabilities by expectations ($\mu(A) \to \mu(I_A)$), random variables by crisp observables ($f \to X_f$).

Definition 3.5. A *state* on $\mathcal{E}(\Omega, \mathcal{F})$ is a map $s : \mathcal{E}(\Omega, \mathcal{F}) \to [0, 1]$ that satisfies $s(1_\Omega) = 1$ and if (f_i) is a sequence in $\mathcal{E}$ such that $\Sigma f_i \in \mathcal{E}(\Omega, \mathcal{F})$, then $s(\Sigma f_i) = \Sigma s(f_i)$.

A state s corresponds to a condition or preparation of a system and

$s(f)$ is interpreted as the probability that the effect f occurs when the system is in the condition corresponding to s. If μ is a probability measure on $(\Omega, \mathcal{F})$, then it follows from the monotone convergence theorem that $\mu : \mathcal{E}(\Omega, \mathcal{F}) \to [0, 1]$ is a state.

Definition 3.6. $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a *σ-morphism* if $\widetilde{X}(1_\Omega) = 1_\Lambda$ and if (f_i) is a sequence in $\mathcal{E}$ such that $\Sigma f_i \in \mathcal{E}(\Omega, \mathcal{F})$, then $\widetilde{X}(\Sigma f_i) = \Sigma \widetilde{X}(f_i)$.

Example 3.7. Let $\Omega = [0, 1]$ and $\Lambda = [1, 2]$. Let $\mathcal{F}$ and $\mathcal{B}$ be the σ-fields of Ω and Λ, respectively. Define $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ by

$$\widetilde{X}(f)(x) = f(x - 1).$$

Then $\widetilde{X}$ is a σ-morphism. In fact, $\widetilde{X}(1_\Omega)(x) = 1_\Omega(x - 1) = 1_\Lambda(x)$ and

$$\widetilde{X}(\Sigma f_i)(x) = \Sigma f_i(x - 1) = \Sigma \widetilde{X}(f_i)(x).$$

Example 3.8. Let $\Omega = \Lambda = [0, 1]$. Let $\mathcal{F}$ and $\mathcal{B}$ be the σ-fields of Ω and Λ, respectively. Define $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ by

$$\widetilde{X}(f)(x) = \frac{1}{2}(f(x) + f(1 - x)).$$

Then $\widetilde{X}$ is a σ-morphism. In fact,

$$\widetilde{X}(1_\Omega)(x) = \frac{1}{2}(1_\Omega(x) + 1_\Omega(1 - x)) = 1_\Lambda(x)$$

and

$$
\begin{aligned}
\widetilde{X}(\Sigma f_i)(x) &= \frac{1}{2}(\Sigma f_i(x) + \Sigma f_i(1-x)) \\
&= \Sigma \frac{1}{2}(f_i(x) + f_i(1-x)) \\
&= \Sigma \widetilde{X}(f_i)(x).
\end{aligned}
$$

Theorem 3.9. ([2]) We have the followings.

(1) If $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \longrightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then $\widetilde{X}(\lambda f) = \lambda \widetilde{X}(f)$ for every $\lambda \in [0,1]$ and $f \in \mathcal{E}(\Omega, \mathcal{F})$,

(2) If $s : \mathcal{E}(\Omega, \mathcal{F}) \to [0,1]$ is a state, then there exists a unique probability measure μ on $(\Omega, \mathcal{F})$ such that $s(f) = \int f d\mu$ for every $f \in \mathcal{E}(\Omega, \mathcal{F})$.

The next result shows that there exists a natural one-to-one correspondence between observables and σ-morphisms.

Theorem 3.10. ([2]) If $X : \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is an observable, then X has a unique extension to a σ-morphism $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$. If $Y : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then $Y|_{\mathcal{F}}$ is an observable.

If $f : \Lambda \to \Omega$ is a measurable function, the corresponding crisp observable $X_f : \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$. The next result shows that $\widetilde{X}_f : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ has a simple form.

Corollary 3.11. ([2]) If $f : \Lambda \to \Omega$ is a measurable function, then $\widetilde{X}_f(g) = g \circ f$ for every $g \in \mathcal{E}(\Omega, F)$, where $\widetilde{X}_f$ is an extension of X_f in Example 3.4.

3.2 The continuity of σ-morphism

In section 3.1, we define the concept of effect, observable and σ-morphism as a probability measure on the set of effects. In this section, we prove the some basic properties of σ-morphism and the continuity of σ-morphism.

Theorem 3.12. If $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then

(1) $\widetilde{X}(0_\Omega) = 0_\Lambda$,

(2) $\widetilde{X}\left(\sum_{i=1}^{n} f_i\right) = \sum_{i=1}^{n} \widetilde{X}(f_i)$,

(3) If $f - g \in \mathcal{E}(\Omega, \mathcal{F})$, then $\widetilde{X}(f-g) = \widetilde{X}(f) - \widetilde{X}(g)$. In particular,

$$\widetilde{X}(1_\Omega - g) = 1_\Lambda - \widetilde{X}(g),$$

(4) $f \le g \Rightarrow \widetilde{X}(f) \le \widetilde{X}(g)$,

Proof. (1) Let $f_1 = 1_\Omega,\ f_i = 0_\Omega\ (i \ge 2)$. Since $\widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) = \widetilde{X}(1_\Omega) = 1_\Lambda$ and

$$\sum_{i=1}^{\infty} \widetilde{X}(f_i) = \widetilde{X}(f_1) + \sum_{i=2}^{\infty} \widetilde{X}(f_i) = 1_\Lambda + \sum_{i=2}^{\infty} \widetilde{X}(0_\Omega),$$

we have $1_\Lambda = 1_\Lambda + \sum_{i=2}^{\infty} \widetilde{X}(0_\Omega)$ and thus $\widetilde{X}(0_\Omega) = 0_\Lambda$.

(2) Let $f_i = 0_\Omega\ (i \ge n+1)$, then $\sum_{i=1}^{\infty} f_i = \sum_{i=1}^{n} f_i$. Thus

$$\widetilde{X}\left(\sum_{i=1}^{n} f_i\right) = \widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} \widetilde{X}(f_i) = \sum_{i=1}^{n} \widetilde{X}(f_i).$$

(3) Since $\widetilde{X}(f) = \widetilde{X}(f-g+g) = \widetilde{X}(f-g) + \widetilde{X}(g)$,

we have

$$\widetilde{X}(f-g)=\widetilde{X}(f)-\widetilde{X}(g).$$

(4) Since $\widetilde{X}(g)=\widetilde{X}(g-f)+\widetilde{X}(f)$, $\widetilde{X}(g-f)\geq 0$.

Theorem 3.13. If $f:(\Lambda_1,\mathcal{B}_1)\to(\Lambda_2,\mathcal{B}_2)$ and $g:(\Lambda_2,\mathcal{B}_2)\to(\Lambda_3,\mathcal{B}_3)$ are measurable functions, then $\widetilde{X}_{g\circ f}=\widetilde{X}_f\circ\widetilde{X}_g$.

Proof. Note that $\widetilde{X}_f:\mathcal{E}(\Lambda_2,\mathcal{B}_2)\to\mathcal{E}(\Lambda_1,\mathcal{B}_1)$, $\widetilde{X}_g:\mathcal{E}(\Lambda_3,\mathcal{B}_3)\to\mathcal{E}(\Lambda_2,\mathcal{B}_2)$ and $\widetilde{X}_{g\circ f}:\mathcal{E}(\Lambda_3,\mathcal{B}_3)\to\mathcal{E}(\Lambda_1,\mathcal{B}_1)$. Since $\widetilde{X}_{g\circ f}(h)(\omega)=h\circ(g\circ f)(\omega)$ for every $h\in\mathcal{E}(\Lambda_3,\mathcal{B}_3)$ and $\omega\in\Lambda_1$,

$$\begin{aligned}\widetilde{X}_{g\circ f}(h)(\omega)&=h\circ(g\circ f)(\omega)=(h\circ g)\circ f(\omega)\\&=\widetilde{X}_f(h\circ g)(\omega)=\widetilde{X}_f\circ\widetilde{X}_g(h)(\omega).\end{aligned}$$

Theorem 3.14. Let $f:\Lambda_2\to\Lambda_1$ be a measurable function and $\mu_i:(\Lambda_i,\mathcal{B}_i)\to[0,1]$ be a probability measure $(i=1,2)$. If $\mu_1=(\mu_2)_f$, then $\mu_2\circ\widetilde{X}_f=\mu_1$.

Proof. Let $g=\sum_{i=1}^{n}c_iI_{B_i}$ be a simple function in $\mathcal{E}(\Lambda_1,\mathcal{B}_1)$. Then by Corollary 3.11,

$$\begin{aligned}\mu_2\circ\widetilde{X}_f(g)&=\int\widetilde{X}_f(g)\,d\mu_2=\int(g\circ f)d\mu_2\\&=\int\sum_{i=1}^{n}(c_iI_{B_i}\circ f)\,d\mu_2\\&=\int\sum_{i=1}^{n}c_iI_{f^{-1}(B_i)}d\mu_2.\end{aligned}$$

And, by the definition of expectation and distribution,

$$\begin{aligned}\int \sum_{i=1}^{n} c_i I_{f^{-1}(B_i)} d\mu_2 &= \sum_{i=1}^{n} c_i \mu_2(f^{-1}(B_i)) \\ &= \sum_{i=1}^{n} c_i (\mu_2)_f(B_i) \\ &= \sum_{i=1}^{n} c_i \mu_1(B_i) \\ &= \mu_1(g).\end{aligned}$$

Hence, $\mu_2 \circ \widetilde{X}_f(g) = \mu_1(g)$.

Now for an arbitrary $g \in \mathcal{E}(\Lambda_1, \mathcal{B}_1)$, there exists an increasing sequence of simple functions $g_n \in \mathcal{E}(\Lambda_1, \mathcal{B}_1)$ such that $\lim_{n\to\infty} g_n = g$. Then by Corollary 3.11,

$$\begin{aligned}\mu_2 \circ \widetilde{X}_f(g) &= \int \widetilde{X}_f(g)\ d\mu_2 \\ &= \int \widetilde{X}_f\left(\lim_{n\to\infty} g_n\right)\ d\mu_2 \\ &= \int \left(\lim_{n\to\infty} g_n \circ f\right) d\mu_2.\end{aligned}$$

By the monotone convergence theorem and the continuity of probability,

$$\begin{aligned}\int \left(\lim_{n\to\infty} g_n \circ f\right) d\mu_2 &= \lim_{n\to\infty} \int (g_n \circ f) d\mu_2 \\ &= \lim_{n\to\infty} \mu_2 \circ \widetilde{X}_f(g_n) \\ &= \lim_{n\to\infty} \mu_1(g_n) \\ &= \mu_1\left(\lim_{n\to\infty} g_n\right) \\ &= \mu_1(g).\end{aligned}$$

Therefore, $\quad \mu_2 \circ \widetilde{X}_f = \mu_1$.

Theorem 3.15. Let $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is an increasing sequence in $\mathcal{E}(\Omega, \mathcal{F})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Proof. Let $f_1 = g_1$ and $f_n = g_n - g_{n-1}$ $(n \geq 2)$. Then $f_n \in \mathcal{E}(\Omega, \mathcal{F})$ for all n and $g_n = \sum_{i=1}^{n} f_i$. Since $g = \sum_{i=1}^{\infty} f_i$, we have

$$\begin{aligned}
\widetilde{X}(g) &= \widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) \\
&= \sum_{i=1}^{\infty} \widetilde{X}(f_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{X}(f_i) \\
&= \lim_{n\to\infty} \widetilde{X}\left(\sum_{i=1}^{n} f_i\right) \\
&= \lim_{n\to\infty} \widetilde{X}(g_n).
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g).$$

Corollary 3.16. Let $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is a decreasing sequence in $\mathcal{E}(\Omega, \mathcal{F})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Theorem 3.17. Let $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is a sequence in $\mathcal{E}(\Omega, \mathcal{F})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Proof. First, we prove that

$$\begin{aligned}\widetilde{X}\left(\liminf_{n\to\infty} g_n\right) &\le \liminf_{n\to\infty} \widetilde{X}(g_n)\\ &\le \limsup_{n\to\infty} \widetilde{X}(g_n)\\ &\le \widetilde{X}\left(\limsup_{n\to\infty} g_n\right).\end{aligned}$$

Let $f_n = \inf_{i\ge n} g_i$. Since (f_n) is an increasing sequence in $\mathcal{E}(\Omega,\mathcal{F})$, by Theorem 3.15, we have

$$\begin{aligned}\widetilde{X}\left(\liminf_{n\to\infty} g_n\right) &= \widetilde{X}\left(\sup_{n\ge 1}\inf_{i\ge n} g_i\right)\\ &= \widetilde{X}\left(\sup_{n\ge 1} f_n\right)\\ &= \widetilde{X}\left(\lim_{n\to\infty} f_n\right)\\ &= \lim_{n\to\infty} \widetilde{X}(f_n).\end{aligned}$$

Let $n \in N$. Then, for each $n \le i$, $f_n \le g_i$, we have $\widetilde{X}(f_n) \le \widetilde{X}(g_i)$ and hence $\widetilde{X}(f_n) \le \inf_{i\ge n} \widetilde{X}(g_i)$. Therefore

$$\sup_{n\ge 1} \widetilde{X}(f_n) \le \sup_{n\ge 1}\inf_{i\ge n} \widetilde{X}(g_i) = \liminf_{n\to\infty} \widetilde{X}(g_n).$$

But, since

$$\lim_{n\to\infty} \widetilde{X}(f_n) = \sup_{n\ge 1} \widetilde{X}(f_n),$$

we have

$$\widetilde{X}\left(\liminf_{n\to\infty} g_n\right) \le \liminf_{n\to\infty} \widetilde{X}(g_n).$$

Similarly, $\limsup_{n\to\infty} \widetilde{X}(g_n) \le \widetilde{X}\left(\limsup_{n\to\infty} g_n\right)$. For $g_n \in \mathcal{E}(\Omega, \mathcal{F})$, since

$$\begin{aligned}\limsup_{n\to\infty} \widetilde{X}(g_n) &\le \widetilde{X}\left(\limsup_{n\to\infty} g_n\right)\\ &= \widetilde{X}\left(\lim_{n\to\infty} g_n\right)\\ &= \widetilde{X}\left(\liminf_{n\to\infty} g_n\right)\\ &\le \liminf_{n\to\infty} \widetilde{X}(g_n),\end{aligned}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \widetilde{X}(g_n) &= \liminf_{n\to\infty} \widetilde{X}(g_n)\\ &= \limsup_{n\to\infty} \widetilde{X}(g_n)\\ &= \widetilde{X}\left(\lim_{n\to\infty} g_n\right)\\ &= \widetilde{X}(g).\end{aligned}$$

Hence

$$\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g).$$

4. The mean value and variance of one-sided fuzzy sets

In this chapter, we define the one-sided fuzzy set and the mean value and variance, defined by Christer Carlsson and Robert Fullér, for various types of one-sided fuzzy sets.

4.1 One-sided fuzzy set

The triangular fuzzy set and quadratic fuzzy set have a continuous membership function. To develop our calculations, we define the new fuzzy sets having discontinuous membership functions.

Definition 4.1. A *left fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a-\alpha, a < x, \\ f(x) & \text{if} \quad a-\alpha \le x \le a, \end{cases}$$

where $f(x)$ is a continuous and increasing function with $f(a-\alpha) = 0$ and $f(a) = 1$. Similarly, a *right fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a, a+\beta < x, \\ g(x) & \text{if} \quad a \le x \le a+\beta, \end{cases}$$

where $g(x)$ is a continuous and decreasing function with $g(a) = 1$ and $g(a+\beta) = 0$. We call these sets one-sided fuzzy sets.

By using this definition, we can define the left triangular fuzzy set as follows.

Definition 4.2. A left triangular fuzzy set is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a - \alpha, a < x, \\ (x - a + \alpha)/\alpha & \text{if} \quad a - \alpha \le x \le a. \end{cases}$$

Similarly, a right triangular fuzzy set can be defined.

By the same ways, left(right) quadratic fuzzy set can be defined. And, to denote these one-sided fuzzy sets, we will use the notations in Definition 2.5 and 2.6. For example, the left triangular fuzzy set in Definition 4.2 is denoted by $(\alpha, a, 0)$.

Remark. Let A be a triangular fuzzy set. Then $A = B + C$ where B be a left triangular fuzzy set and C be a right triangular fuzzy set.

Theorem 4.3. (1) Let A_1 and A_2 be the left fuzzy sets. Then $A_1 + A_2$ is a left fuzzy set.
(2) Let A_1 and A_2 be the right fuzzy sets. Then $A_1 + A_2$ is a right fuzzy set.

Proof. (1) Let A_1 and A_2 be the left fuzzy sets with membership functions

$$\mu_{A_1}(x) = \begin{cases} 0 & \text{if} \quad x < a_1 - \alpha_1, a_1 < x, \\ f_1(x) & \text{if} \quad a_1 - \alpha_1 \le x \le a_1. \end{cases}$$

and

$$\mu_{A_2}(x) = \begin{cases} 0 & \text{if} \quad x < a_2 - \alpha_2, a_2 < x, \\ f_2(x) & \text{if} \quad a_2 - \alpha_2 \le x \le a_2. \end{cases}$$

respectively, where $f_1(x)$ and $f_2(x)$ are continuous and increasing functions with $f_1(a_1 - \alpha_1) = 0$, $f_1(a_1) = 1$, $f_2(a_2 - \alpha_2) = 0$ and $f_2(a_2) = 1$. Since

$[A_1]^\gamma = [f_1^{-1}(\gamma), a_1]$ and $[A_2]^\gamma = [f_2^{-1}(\gamma), a_2]$, we have $[A_1 + A_2]^\gamma = [f_1^{-1}(\gamma) + f_2^{-1}(\gamma), a_1 + a_2]$. Hence $A_1 + A_2$ is a left fuzzy set.

(2) By the similar manner, (2) can be obtained.

Example 4.4. Let $A_1 = (2, 4, 0)$ and $A_2 = (3, 7, 0)$ be the left triangular fuzzy sets and $B_1 = (0, 5, 3)$ and $B_2 = (0, 7, 2)$ be the right triangular fuzzy sets. Let $[A_1]^\gamma$, $[A_2]^\gamma$, $[B_1]^\gamma$ and $[B_2]^\gamma$ be the γ-level sets of A_1, A_2, B_1 and B_2, respectively.

(1) Since $[A_1]^\gamma = [2\gamma + 2, 4]$ and $[A_2]^\gamma = [3\gamma + 4, 7]$, we have $[A_1 + A_2]^\gamma = [5\gamma + 6, 11]$. Hence $A_1 + A_2 = (5, 11, 0)$ is a left triangular fuzzy set,

(2) $[B_1]^\gamma = [5, 8 - 3\gamma]$ and $[B_2]^\gamma = [7, 9 - 2\gamma]$, we have $[B_1 + B_2]^\gamma = [12, 17 - 5\gamma]$. Hence $B_1 + B_2 = (0, 12, 5)$ is a right triangular fuzzy set.

Example 4.5. Let $A_1 = [1, 2, 0]$ and $A_2 = [3, 4, 0]$ be the left quadratic fuzzy sets and $B_1 = [0, 5, 8]$ and $B_2 = [0, 2, 3]$ be the right quadratic fuzzy sets. Let $[A_1]^\gamma$, $[A_2]^\gamma$,$[B_1]^\gamma$ and $[B_2]^\gamma$ be the γ-level sets of A_1, A_2, B_1 and B_2, respectively.

(1) Since $[A_1]^\gamma = [2 - \sqrt{1-\gamma}, 2]$ and $[A_2]^\gamma = [4 - \sqrt{1-\gamma}, 4]$, we have $[A_1 + A_2]^\gamma = [6 - 2\sqrt{1-\gamma}, 6]$. Hence $A_1 + A_2 = [4, 6, 0]$ is a left quadratic fuzzy set,

(2) Since $[B_1]^\gamma = [5, 5 + 3\sqrt{1-\gamma}]$ and $[B_2]^\gamma = [2, 2 + \sqrt{1-\gamma}]$, we have $[B_1+B_2]^\gamma = [7, 7+4\sqrt{1-\gamma}]$. Hence $B_1+B_2 = [0, 7, 11]$ is a right quadratic fuzzy set.

Remark. Let A and B be the left fuzzy set and right fuzzy set, respectively.

Then $A+B$ may not be an one-sided fuzzy set.

Example 4.6. Let $A=(2,4,0)$ be a left triangular fuzzy set and $B=(0,5,3)$ be a right triangular fuzzy set. Let $[A]^\gamma$ and $[B]^\gamma$ be the γ-level sets of A and B, respectively. Since $[A]^\gamma=[2\gamma+2,4]$ and $[B]^\gamma=[5,8-3\gamma]$, we have $[A+B]^\gamma=[2\gamma+7,12-3\gamma]$. Thus $A+B=(2,9,3)$ is a triangular fuzzy set, but it is not an one-sided fuzzy set.

Example 4.7. Let $A=[1,2,0]$ be a left quadratic fuzzy set and $B=[0,5,8]$ be a right quadratic fuzzy set. Let $[A]^\gamma$ and $[B]^\gamma$ be the γ-level sets of A and B, respectively. Since $[A]^\gamma=[2-\sqrt{1-\gamma},2]$ and $[B]^\gamma=[5,5+3\sqrt{1-\gamma}]$, we have $[A+B]^\gamma=[7-\sqrt{1-\gamma},7+3\sqrt{1-\gamma}]$. Thus $A+B$ is neither an one-sided fuzzy set nor a quadratic fuzzy set.

4.2 The mean value and variance of one-sided fuzzy sets

In this section, we introduce the notion of possibilistic mean value and variance of fuzzy sets defined by C. Carlsson and R. Fullér. And, we calculate the possibilistic mean value and variance of one-sided fuzzy sets.

Definition 4.8. Let A be a fuzzy set with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$. The *lower possibilistic mean value* of A is defined by

$$M_*(A) = 2\int_0^1 \gamma a_1(\gamma) d\gamma.$$

Similarly, the *upper possibilistic mean value* of A is defined by

$$M^*(A) = 2\int_0^1 \gamma a_2(\gamma) d\gamma.$$

Note that $M_*(A)$ is the lower possibility-weighted average of the minima of the γ-sets and $M^*(A)$ is the upper possibility-weighted average of the maxima of the γ-sets.

Definition 4.9. Let A be a fuzzy set. The *interval-valued possibilistic mean* of A is defined by

$$M(A) = [M_*(A), M^*(A)].$$

Theorem 4.10. ([7]) Let A and B be two non-interactive fuzzy sets and let $\lambda \in \mathbb{R}$. Then

$$M(A+B) = M(A) + M(B), \;\; M(\lambda A) = \lambda M(A).$$

Definition 4.11. The *crisp possibilistic mean value* of a fuzzy set A is the arithmetic mean of its lower possibilistic and upper possibilistic mean values, i.e.,

$$\overline{M}(A) = \frac{M_*(A) + M^*(A)}{2}.$$

Theorem 4.12. ([7]) Let A and B be the fuzzy sets and $\lambda \in \mathbb{R}$. Then

$$\overline{M}(A+B) = \overline{M}(A) + \overline{M}(B),\ \overline{M}(\lambda A) = \lambda \overline{M}(A).$$

Example 4.13. Let $A = (\alpha, a, 0)$ be a left triangular fuzzy set. Then a γ-level set of A is

$$[A]^\gamma = [a - (1-\gamma)\alpha, a], \gamma \in [0,1].$$

Thus

$$M_*(A) = 2\int_0^1 \gamma(a - (1-\gamma)\alpha)d\gamma = a - \frac{\alpha}{3}$$

and

$$M^*(A) = 2\int_0^1 \gamma \cdot a d\gamma = a.$$

Hence we have

$$M(A) = \left[a - \frac{\alpha}{3}, a\right]$$

and

$$\overline{M}(A) = \int_0^1 \gamma(a - (1-\gamma)\alpha + a)d\gamma = a - \frac{\alpha}{6}.$$

The following theorems show the relations between $M_*(AB)$ and $M_*(A)M_*(B)$ and between $M^*(AB)$ and $M^*(A)M^*(B)$ for triangular fuzzy sets and quadratic fuzzy sets.

Theorem 4.14. Let $A = (\alpha_1, a, \beta_1)$ and $B = (\alpha_2, b, \beta_2)$ be the triangular fuzzy sets. Then $M_*(A)M_*(B) \leq M_*(AB)$ and $M^*(A)M^*(B) \leq M^*(AB)$.

Proof. Since $[A]^\gamma = [a-(1-\gamma)\alpha_1, a+(1-\gamma)\beta_1]$ and $[B]^\gamma = [b-(1-\gamma)\alpha_2, b+(1-\gamma)\beta_2]$, we have

$$M_*(A) = 2\int_0^1 \gamma(a-(1-\gamma)\alpha_1)d\gamma = a - \frac{\alpha_1}{3}$$

and

$$M_*(B) = 2\int_0^1 \gamma(b-(1-\gamma)\alpha_2)d\gamma = b - \frac{\alpha_2}{3}.$$

Since

$$\begin{aligned} M_*(AB) &= 2\int_0^1 \gamma(ab - a(1-\gamma)\alpha_2 - b(1-\gamma)\alpha_1 + (1-\gamma)^2\alpha_1\alpha_2)d\gamma \\ &= ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{6}, \end{aligned}$$

we have

$$\begin{aligned} M_*(A)M_*(B) &= \left(a - \frac{\alpha_1}{3}\right)\left(b - \frac{\alpha_2}{3}\right) \\ &= ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{9} \\ &\leq ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{6} \\ &= M_*(AB). \end{aligned}$$

Similarly, we can show that $M^*(A)M^*(B) \leq M^*(AB)$.

Theorem 4.15. Let $A = [\alpha_1, k_1, \beta_1]$ and $B = [\alpha_2, k_2, \beta_2]$ be the quadratic fuzzy sets. Then $M_*(A)M_*(B) \leq M_*(AB)$ and $M^*(A)M^*(B) \leq M^*(AB)$.

Proof. Let $a = \frac{4}{(\alpha_1-\beta_1)^2}$ and $b = \frac{4}{(\alpha_2-\beta_2)^2}$. Then

$$[A]^\gamma = \left[k_1 - \sqrt{\frac{1-\gamma}{a}}, k_1 + \sqrt{\frac{1-\gamma}{a}}\right]$$

and

$$[B]^\gamma = \left[k_2 - \sqrt{\frac{1-\gamma}{b}}, k_2 + \sqrt{\frac{1-\gamma}{b}}\right].$$

Thus

$$M_*(A) = 2\int_0^1 \gamma\left(k_1 - \sqrt{\frac{1-\gamma}{a}}\right)d\gamma = k_1 - \frac{8}{15\sqrt{a}}$$

and

$$M_*(B) = 2\int_0^1 \gamma\left(k_2 - \sqrt{\frac{1-\gamma}{b}}\right)d\gamma = k_2 - \frac{8}{15\sqrt{b}}.$$

Since

$$\begin{aligned}M_*(AB) &= 2\int_0^1 \gamma\left(k_1k_2 - k_1\sqrt{\frac{1-\gamma}{b}} - k_2\sqrt{\frac{1-\gamma}{a}} + \frac{1-\gamma}{\sqrt{ab}}\right)d\gamma \\ &= k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{1}{3\sqrt{ab}},\end{aligned}$$

we have

$$\begin{aligned}M_*(A)M_*(B) &= \left(k_1 - \frac{8}{15\sqrt{a}}\right)\left(k_2 - \frac{8}{15\sqrt{b}}\right) \\ &= k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{64}{225\sqrt{ab}} \\ &\leq k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{1}{3\sqrt{ab}} \\ &= M_*(AB).\end{aligned}$$

Similarly, we can show that $M^*(A)M^*(B) \leq M^*(AB)$.

Theorem 4.16. Let A be a left fuzzy set with $[A]^\gamma = [a_1(\gamma), a]$ and B be a right fuzzy set with $[B]^\gamma = [b, b_2(\gamma)]$, then $M(AB) = M(A)M(B)$.

Proof. Since $[A]^\gamma = [a_1(\gamma), a]$ and $[B]^\gamma = [b, b_2(\gamma)]$, we have

$$M_*(A) = 2\int_0^1 \gamma a_1(\gamma) d\gamma$$

and

$$M^*(A) = 2\int_0^1 \gamma \cdot a d\gamma = a.$$

Similarly, we have $M_*(B) = b$ and $M^*(B) = 2\int_0^1 \gamma b_2(\gamma)d\gamma$. Thus

$$M_*(A)M_*(B) = \left(2\int_0^1 \gamma a_1(\gamma)d\gamma\right)\cdot b = 2b\int_0^1 \gamma a_1(\gamma)d\gamma$$

and

$$M^*(A)M^*(B) = a\cdot\left(2\int_0^1 \gamma b_2(\gamma)d\gamma\right) = 2a\int_0^1 \gamma b_2(\gamma)d\gamma.$$

Since $[AB]^\gamma = [ba_1(\gamma), ab_2(\gamma)]$, we have

$$M_*(AB) = 2b\int_0^1 \gamma a_1(\gamma)d\gamma$$

and

$$M^*(AB) = 2a\int_0^1 \gamma b_2(\gamma)d\gamma.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, $M(AB) = M(A)M(B)$.

Theorem 4.16 can be considered as the similar result which is well-known in the independence of events in probability theory.

Example 4.17. Let A be a left triangular fuzzy set with $[A]^\gamma = [2\gamma+2, 4]$ and B be a right triangular fuzzy set with $[B]^\gamma = [7, 9-2\gamma]$. Then

$$M_*(A) = 2\int_0^1 \gamma(2\gamma+2)d\gamma = \frac{10}{3}$$

and

$$M^*(A) = 2\int_0^1 4\gamma d\gamma = 4.$$

Similarly, we have $M_*(B) = 7$ and $M^*(B) = \frac{23}{3}$. Thus $M_*(A)M_*(B) = \frac{70}{3}$ and $M^*(A)M^*(B) = \frac{92}{3}$. Since $[AB]^\gamma = [14\gamma+14, 36-8\gamma]$, we have

$$M_*(AB) = 2\int_0^1 \gamma(14\gamma+14)d\gamma = \frac{70}{3}$$

and

$$M^*(AB) = 2\int_0^1 \gamma(36-8\gamma)d\gamma = \frac{92}{3}.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, we have $M(AB) = M(A)M(B)$.

Example 4.18. Let A be a left quadratic fuzzy set with $[A]^\gamma = [2-\sqrt{1-\gamma}, 2]$ and B be a right quadratic fuzzy set with $[B]^\gamma = [5, 5+3\sqrt{1-\gamma}]$. Then

$$M_*(A) = 2\int_0^1 \gamma(2-\sqrt{1-\gamma})d\gamma = \frac{22}{15}$$

and

$$M^*(A) = 2\int_0^1 2\gamma d\gamma = 2.$$

Similarly, we have $M_*(B) = 5$ and $M^*(B) = \frac{33}{5}$. Thus $M_*(A)M_*(B) = \frac{22}{3}$ and $M^*(A)M^*(B) = \frac{66}{5}$. Since $[AB]^\gamma = [10-5\sqrt{1-\gamma}, 10+6\sqrt{1-\gamma}]$, we have

$$M_*(AB) = 2\int_0^1 \gamma(10 - 5\sqrt{1-\gamma})d\gamma = \frac{22}{3}$$

and

$$M^*(AB) = 2\int_0^1 \gamma(10 + 6\sqrt{1-\gamma})d\gamma = \frac{66}{5}.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, we have

$$M(A)M(B) = M(AB).$$

Remark. If A and B are left fuzzy sets (or right fuzzy sets), the equality $M(A)M(B) = M(AB)$ does not always hold.

Definition 4.19. The *variance of fuzzy set* A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ is defined by

$$Var(A) = \frac{1}{2}\int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))^2 d\gamma$$

Example 4.20. If $A = (\alpha, a, 0)$ is a left triangular fuzzy set then

$$Var(A) = \frac{1}{2}\int_0^1 \gamma(a - (a - \alpha(1-\gamma)))^2 d\gamma = \frac{\alpha^2}{24}.$$

Definition 4.21. The *covariance between fuzzy sets* A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and B with $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$) is defined by

$$Cov(A, B) = \frac{1}{2}\int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))(b_2(\gamma) - b_1(\gamma))d\gamma.$$

Proposition 4.22. Let $A_1 = (\alpha_1, a_1, 0)$ and $A_2 = (\alpha_2, a_2, 0)$ be the left triangular fuzzy sets and $B_1 = (0, b_1, \beta_1)$ and $B_2 = (0, b_2, \beta_2)$ be the right triangular fuzzy sets. Then

(1) $Cov(A_1, A_2) = \dfrac{\alpha_1\alpha_2}{24}$.

(2) $Cov(B_1, B_2) = \dfrac{\beta_1\beta_2}{24}$.

(3) $Cov(A_1, B_2) = \dfrac{\alpha_1\beta_2}{24}$.

Proof. (1) Since $[A_1]^\gamma = [a_1 - (1-\gamma)\alpha_1, a_1]$ and $[A_2]^\gamma = [a_2 - (1-\gamma)\alpha_2, a_2]$, we have

$$Cov(A_1, A_2) = \frac{1}{2}\int_0^1 \alpha_1\alpha_2\gamma(1-\gamma)^2 d\gamma = \frac{\alpha_1\alpha_2}{24}.$$

By the similar method, (2) and (3) can be obtained.

Example 4.23. Let $A_1 = (2, 4, 0)$ and $A_2 = (3, 7, 0)$ be the left triangular fuzzy sets and $B_1 = (0, 5, 3)$ and $B_2 = (0, 7, 2)$ be the right triangular fuzzy sets.

(1) Since $[A_1]^\gamma = [2\gamma + 2, 4]$ and $[A_2]^\gamma = [3\gamma + 4, 7]$, $Cov(A_1, A_2) = \frac{1}{4}$.

(2) Since $[B_1]^\gamma = [5, 8 - 3\gamma]$ and $[B_2]^\gamma = [7, 9 - 2\gamma]$, $Cov(B_1, B_2) = \frac{1}{4}$.

(3) Since $[A_1]^\gamma = [2\gamma + 2, 4]$ and $[B_2]^\gamma = [7, 9 - 2\gamma]$, $Cov(A_1, B_2) = \frac{1}{6}$.

Proposition 4.24. Let $A_1 = [\alpha_1, k_1, 0]$ and $A_2 = [\alpha_2, k_2, 0]$ be the left quadratic fuzzy sets and $B_1 = [0, m_1, \beta_1]$ and $B_2 = [0, m_2, \beta_2]$ be the right quadratic fuzzy sets. And put $a_1 = \frac{1}{(\alpha_1 - k_1)^2}$, $a_2 = \frac{1}{(\alpha_2 - k_2)^2}$, $b_1 = \frac{1}{(\beta_1 - m_1)^2}$ and $b_2 = \frac{1}{(\beta_2 - m_2)^2}$. Then

(1) $Cov(A_1, A_2) = \dfrac{1}{12\sqrt{a_1 a_2}}$.

(2) $Cov(B_1, B_2) = \dfrac{1}{12\sqrt{b_1 b_2}}$.

(3) $Cov(A_1, B_2) = \dfrac{1}{12\sqrt{a_1 b_2}}$.

Proof. (1) Since

$$[A_1]^\gamma = \left[k_1 - \sqrt{\frac{1-\gamma}{a_1}}, k_1\right], \quad [A_2]^\gamma = \left[k_2 - \sqrt{\frac{1-\gamma}{a_2}}, k_2\right],$$

we have

$$Cov(A_1, A_2) = \frac{1}{2}\int_0^1 \gamma \frac{1-\gamma}{\sqrt{a_1 a_2}} d\gamma = \frac{1}{12\sqrt{a_1 a_2}}.$$

By the similar method, (2) and (3) can be obtained.

Example 4.25. Let $A_1 = [1, 2, 0]$ and $A_2 = [3, 4, 0]$ be the left quadratic fuzzy sets and $B_1 = [0, 5, 8]$ and $B_2 = [0, 2, 3]$ be the right quadratic fuzzy sets.

(1)Since $[A_1]^\gamma = [2-\sqrt{1-\gamma}, 2]$ and $[A_2]^\gamma = [4-\sqrt{1-\gamma}, 4]$, $Cov(A_1, A_2) = \frac{1}{12}$.

(2)Since $[A_1]^\gamma = [2-\sqrt{1-\gamma}, 2]$ and $[B_2]^\gamma = [2, 2+\sqrt{1-\gamma}]$, $Cov(A_1, B_2) = \frac{1}{12}$.

(3)Since $[A_2]^\gamma = [4-\sqrt{1-\gamma}, 4]$ and $[B_1]^\gamma = [5, 5+3\sqrt{1-\gamma}]$, $Cov(A_2, B_1) = \frac{1}{4}$.

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<국문초록>

이펙트에 대한 研究

본 논문에서는 사건과 확률변수의 일반화로서 각각 이펙트(또는 퍼지사건)과 가관찰량(또는 퍼지확률변수)를 소개하였다. 아울러, 이팩트들의 집합 위에서 확률측도로서의 시그마-모피즘을 소개하였고 그것의 연속성과 몇가지 기본적인 성질들을 증명하였다.

퍼지 집합들을 위한 다양한 형태의 평균과 분산들이 존재한다. 본 논문에서는 크리스터 칼슨과 로버트 펄러에 의해 정의된 평균과 분산을 연구하였다. 서로 독립인 확률변수 A와 B에 대하여 $M(AB)=M(A)M(B)$가 성립한다. 그리고 한쪽-방면 퍼지 집합들의 독립성에 대한 성질을 탐구하였고 두 개의 한쪽-방면 퍼지 집합 A와 B에 대하여

$$M(AB)=M(A)M(B)$$

가 성립하는 것을 증명하였다. 또한 한쪽-방면 퍼지 집합들과 그들의 곱집합에 대한 파서블릭 평균값, 분산 그리고 공분산을 계산하였다.

감사의 글

학위 과정의 길은 참 배움의 길 보다는 자신과의 약속을 지키기 위해 노력했던 시간들의 모임에 더 가깝지 않았나 저는 생각하고 싶습니다. 비로소 이제 모든 과정을 마치고 마지막 갈무리를 한 장의 글로 표현하려 하니 가진 걸 나누지 못하고 늘 받기만 했던 삶을 반성하게 됩니다. 먼저 박사 학위를 취득할 수 있도록 도움을 주신 많은 분들께 글로서 나마 감사의 마음을 전해 드리고자 합니다.

많이 부족한 저를 배움의 길로 불러 오늘의 모습으로 있게 해 주신 지도교수 윤용식 교수님께 머리 숙여 깊히 감사를 드립니다. 또한 논문 심사과정에서 세심히 지적해주시고 지도해주신 고윤희 교수님, 정승달 교수님, 진현성 교수님과 수학에 재미를 느끼도록 해주셨고 수학하는 자세를 일깨워 주신 박진원 교수님, 그 외 수학과, 수학교육과 교수님들께도 감사 드립니다. 그리고 함께 공부하며 서로에게 힘이 되어 주셨던 선배님들을 비롯한 후배님들께도 고마운 마음을 전하고 싶습니다.

인생의 처음 날 부터 늘 사랑과 희생으로 든든한 울타리가 되어 주신 부모님, 철없는 막내 사위의 모든 것을 이해해 주시고 감싸 주시던 장모님 그리고 하늘에서 따뜻한 미소로 지켜보고 계실 장인 어른께 이 논문을 바칩니다.

마지막으로, 세상 아니 우주에서 가장 소중한 아내, 변복실. 그녀의 자상한 배려가 없었으면 지금의 나는 없었을 것입니다. 직장 일을 하면서도 사랑스런 두 아이를 건강하게 키워내고 있는 아내, 내 아내에게 활기찬 성원을 흠뻑 전하고 싶고, 우리 부부에게 너무 소중한 보물인 서연이, 예진이에게 늘 자랑스런 아빠가 될 수 있도록 더욱 정진하겠습니다.

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