博士學位論文

Linear preservers of extremes of maximal column rank inequalities over semirings

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Linear preservers of extremes of maximal column rank inequalities over semirings

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半環상에서 極大 列 階數 不等式의 極値를 保存하는 線形演算子

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CONTENTS

<Abstract>

Linear preservers of extremes of maximal column rank inequalities over semirings

During the past century a lot of literature has been devoted to the problems of determining the linear operators on the $m\times n$ matrix algebra $M_{m,n}(F)$ over a field F that leave certain matrix subsets invariant. In 1987, Kantor and Frobenius proved that if a linear operator T on $M_{m,n}(R)$ preserves the determinant of matrices then T has the form $T(X)=UXV$ or $T(X)=UX^tV$. Since these papers was published, many researchers have investigated to characterize the linear operators that preserve certain subsets of $M_{m,n}(F)$. We call these researches as "Linear Preserver Problems", which is an major topic on linear algebra and matrix theory.

In this thesis, we study the inequalities of maximal column rank for the sum and product of two matrices over semirings. There was some papers on the researches of both maximal column rank of one matrix and extremes of factor rank over semirings. We used those papers in order to research the linear operators that preserve the sets of matrix pairs which satisfy the extremes of maximal column rank inequalities. We constitute the sets of matrix pairs which are the extremes of maximal column rank inequalities. We characterize the linear operators that preserve the 7 extreme sets of maximal column rank inequalities. That is, we prove that those linear operators are $(P,Q,B)-operator$ such that $T(X)=P(X\circ B)Q$ or $T(X)=P(X\circ B)^tQ$.

<국문초록>

半環상에서 極大 列 階數 不等式의 極値를 保存하는 線形演算子

지난 100년 동안 여러 수학자들은 $m \times n$ 行列들의 집합 $M_{m,n}(F)$의 부분집합들의 特性과 그 특성을 보존하는 線形演算子 問題에 대하여 연구해 왔다. 1897년 칸토르와 프로베니우스가 行列式의 값을 보존하는 線形演算子가 $T(X) = UXV$ 또는 $T(X) = UX^{t}V$ 형태로 정해짐을 證明한 것을 시작으로, 선형연산자의 형태 糾明과 그의 同値 條件들을 찾는 문제는 많은 연구자들의 研究主題가 되어 왔다. 이 연구는 体, 環, 半環, 부울 대수 등의 다양한 代數的 구조 위에서 "線形保存子 問題"라는 이름으로 線型代數學의 중심과제의 하나가 되어 연구되어 왔다.

본 研究에서는 半環상에서 두 행렬의 합과 곱에 대한 極大 列 階數 不等式을 연구하였다. 이 연구에 앞서 발표된 논문들에서는 한 행렬의 극대 열 계수에 대하여 연구된 바 있고, 두 행렬의 합과 곱에 대한 分解 階數 부등식에 대하여 연구되기도 하였다. 본 연구에서는 이러한 선행연구들을 參考하여, 두 행렬의 합과 곱에 대한 극대 열 계수 부등식을 分析하고, 이 부등식들이 極値가 되게 하는 행렬의 순서쌍 집합들을 구성하였다. 그리고 이 행렬의 순서쌍들로 이루어진 7가지의 極値集合들을 보존하는 선형연산자를 규명하는 문제를 解決하였다. 곧, 이 극치집합들을 보존하는 선형연산자는 $T(X) = P(X \circ B)Q$ 또는 $T(X) = P(X \circ B)^{t}Q$ 형태로서 $(P, Q, B)-operator$로 정하여짐을 證明하였다.

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2006年　12月

1 Introduction

During the past century a lot of literature has been devoted to the problems of determining the linear operators on the $m \times n$ matrix algebra $M_{m\times n}(F)$ (if $m = n$, we use the notation $M_n(F)$) over a field F that leave certain matrix subsets invariant, see [10]. For a survey of these type of problems, see [10]. These problems have been extended to the $m \times n$ matrices over various semirings, see [1, 2].

Marsaglia and Styan studied on the inequalities for the rank of matrices, see [8]. Beasley and Guterman investigated the rank inequalities of matrices over semirings, see [1]. And they characterized the equality cases for some inequalities in [2]. This characterization problems are open even over fields as well as over semirings, see [9]. The structure of matrix varieties which arise as extremal cases in the inequalities is far from being understood over fields, as well as semirings. For the investigation of linear preserver problems of extreme cases of the rank inequalities of matrices over fields was obtained in [4]. A usual way to generate elements of such a variety is to find a matrix pairs which belongs to it and to act on this set by various linear operators that preserve this variety. Beasley and Guterman characterized the linear operators that preserve extremal cases of rank inequalities over semiring, see [2]. Song and his colleagues characterized the linear operators that preserve maximal column rank in [7, 11].

In this thesis, I characterize linear operators that preserve the sets of matrix pairs which satisfy equality cases for the maximal column rank inequalities over semirings.

2 Definitions and Preliminaries

Definition 2.1. A *semiring* $\mathbb{S}$ consists of a set and two binary operations, addition and multiplication, such that:

- $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathbb{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathbb{S}$.

In this paper we will always assume that there is a multiplicative identity 1 in $\mathbb{S}$ which is different from 0.

In particular, a semiring $\mathbb{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

Throughout this paper, we will assume that all semirings are antinegative and have no zero divisors.

Definition 2.2. The *Boolean semiring* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathbb{M}_{m,n}(\mathbb{S})$ denote the set of $m \times n$ matrices with entries from the semiring $\mathbb{S}$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{S})$ instead of $\mathbb{M}_{n,n}(\mathbb{S})$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. Let R_i denote the matrix whose i^{th} row is all ones and all other rows are zero, and C_j denote the matrix whose j^{th} column is all ones and all other columns are zero. Let U_k denote the $k \times k$ matrix of all ones above and on the main diagonal, L_k denote $k \times k$ strictly lower triangular matrix of ones.

The matrix $E_{i,j}$, called a *cell*, denotes the matrix with 1 in (i,j) position and zero elsewhere. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{S}$.

A *line* of a matrix A is a row or a column of A. We let $\mathcal{Z}(\mathbb{S}) = \{x \in \mathbb{S} | xy = yx, \forall y \in \mathbb{S}\}$ denote the *center* of the semiring $\mathbb{S}$, and $|A|$ denote the number of nonzero entries in the matrix A, and $A[i_1, \ldots, i_k | j_1, \ldots, j_l]$ denote the $k \times l$-submatrix of A which lies in the intersection of the $i_1, \ldots, i_k$ rows and $j_1, \ldots, j_l$ columns.

Let $\Delta_{m,n} = \{(i,j) | i = 1, \ldots, m; j = 1, \ldots, n\}$. If $m = n$, we use the notation Δ_n instead of $\Delta_{n,n}$.

We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$ in this case. If A and B are matrices and $A \geq B$ we let $A \setminus B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } \; b_{i,j} = 1 \\ a_{i,j} & \text{otherwise} \end{cases} .$$

Definition 2.3. An element in $\mathbb{M}_{n,1}(\mathbb{S})$ is called a *vector* over $\mathbb{S}$.

A set of vectors with entries from a semiring is called *linearly independent* if there is no vector in this set that can be expressed as a nontrivial linear combination of the others.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *maximal column rank* k ($mc(A) = k$) if k is the maximal number of the columns of A which are linearly independent.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *maximal row rank* k ($mr(A) = k$) if k is the maximal number of the rows of A which are linearly independent.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *factor rank* k ($rank(A) = k$) if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{S})$ and $C \in \mathbb{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and k is the smallest positive integer for which such factorization exists.

Remark 2.4. It follows that

$$1 \leq rank(A) \leq mc(A) \leq n \tag{1.1}$$

for all nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

If $\mathbb{S}$ is a subsemiring of a real field then there is a real rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$, which is considered as a matrix over real field. Easy examples show that over semirings these functions are not equal in general. However, the inequality $mc(A) \geq \rho(A)$ always hold.

Theorem 2.5. [1] *Let $\mathbb{S}$ be an antinegative semiring without zero divisors. If $A, B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $A \neq O, B \neq O$. Then*

1. $1 \leq mc(A + B)$;
2. $mc(A + B) \leq n$.

If $A \in \mathbb{M}_{m,n}(\mathbb{S})$, $B \in \mathbb{M}_{n,k}(\mathbb{S})$ with $A \neq O, B \neq O$. Then

3. *if* $mc(A) + mr(B) > n$ *then* $mc(AB) \geq 1$;
4. $mc(AB) \leq mc(B)$.

If $\mathbb{S}$ is a subsemiring of $\mathbb{R}^+$, the nonnegative reals. Then

5. $mc(A + B) \geq |\rho(A) - \rho(B)|$.

For $A \in \mathbb{M}_{m,n}(\mathbb{S})$, $B \in \mathbb{M}_{n,k}(\mathbb{S})$ one has that

6. *if* $\rho(A) + \rho(B) \leq n$ *then* $mc(AB) \geq 0$;
7. *if* $\rho(A) + \rho(B) > n$ *then* $mc(AB) \geq \rho(A) + \rho(B) - n$.

Proof. 1. This inequality follow directly from the definition of maximal column rank and the condition that $A, B \neq 0$. For the exactness one can take $A = B = E_{1,1}$. Let $\mathbb{B}$ be a Boolean semiring. For each pair (r, s), $0 \leq r, s \leq m$ we consider the matrices $A_r = J \setminus (\sum_{i=1}^{r} E_{i,i})$, $B_s = J \setminus (\sum_{i=1}^{s} E_{i,i+1})$ if $s < m$ and $B_s = J \setminus (\sum_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$ if $s = m$. Then

$$mc(A_r) = r, mc(B_s) = s$$

by definition and $A_r + B_s = J$ has maximal column rank equal to 1. Thus, this bound is the best possible over Boolean semiring.

3. For an arbitrary antinegative semiring, if $mc(A) = i$ and $mr(B) = j$ then A has at least i nonzero columns while B has at least j nonzero rows. Thus, if $i + j > n$, $AB \neq O$ and hence this bound is established. For the proof of exactness let us take $A = B = E_{1,1}$.

Let $\mathbb{B}$ be a Boolean semiring. In the case $m = n = k$ for each pair (r, s), $1 \leq r, s \leq n$ let us consider the matrices $A_r = \sum_{i=1}^{r} E_{i,i} + \sum_{i=1}^{m} E_{i,1}$, $B_s = \sum_{i=1}^{s} E_{i,i} + \sum_{i=1}^{n} E_{1,i}$. Then

$$mc(A_r) = r, mc(B_s) = s$$

by definition and $A_r B_s = J$. Thus $mc(A_r B_s) = 1$. It is routine to generalize this example to the case $m \neq n \neq k$.

4. For the proof that this bound is exact and the best possible, consider

$$A_r = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix} \text{ and } B_s = \begin{bmatrix} I_s & O_{s,n-s} \\ O_{n-s,s} & O_{n-s,n-s} \end{bmatrix}$$

for each pair r, s, $1 \leq r, s \leq n$ in the case $m = n$. It is routine to generalize this example to the case $m \neq n$.

5. This inequality follow directly from the fact that $\rho(X) \leq mc(X)$ for all $X \in \mathbb{M}_{m,n}(\mathbb{R}^+)$, and corresponding inequalities for matrices with coefficients from the field $\mathbb{R}^+$. For the proof of exactness consider matrices $A = E_{1,1} + \ldots +$

$E_{n-1,n-1}$, $B = J \setminus A$. In order to show that this bound is the best possible one can take the family of matrices A_r, B_s,

$$A_r = \begin{bmatrix} L_{r+1} & O_{r+1,n-r-1} \\ O_{m-r-1,r+1} & O_{m-r-1,n-r-1} \end{bmatrix} \quad \text{and} \quad B_s = \begin{bmatrix} U_s & O_{s,n-s} \\ O_{m-s,s} & O_{m-s,n-s} \end{bmatrix}.$$

7. This inequality follows directly from the fact that $\rho(X) \leq mc(X)$ for all $X \in \mathbb{M}_{m,n}(\mathbb{R}^+)$, and corresponding inequalities for matrices with coefficients from the field $\mathbb{R}$. For the exactness one can take $A = B = I$. In order to show that this bound is the best possible one can take the family of matrices A_r, B_s,

$$A_r = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix} \quad \text{and} \quad B_s = \begin{bmatrix} O_{n-s,n-s} & O_{n-s,s} \\ O_{s,n-s} & I_s \end{bmatrix}.$$

$\blacksquare$

The following examples shows that $mc(A+B) \not\leq mc(A)+mc(B)$, $mc(AB) \not\leq min\{mc(A), mc(B)\}$ which is different from the rank inequality of the matrices over real field.

Example 2.6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+)$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+)$, where $\mathbb{Z}^+$ is the semiring of nonnegative integers. Then $mc(A) = 1$, $mc(B) = 1$, and $mc(A + B) = 3$ over $\mathbb{Z}^+$.

Example 2.7. Let $A = \begin{bmatrix} 3 & 7 & 7 \end{bmatrix} \in \mathbb{M}_{1,3}(\mathbb{Z}^+)$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+)$, where $\mathbb{Z}^+$ is the semiring of nonnegative integers. Then $mc(A) = 2$, $mc(B) = 3$, and $mc(AB) = mc(\begin{bmatrix} 3 & 10 & 17 \end{bmatrix}) = 3$ over $\mathbb{Z}^+$.

Definition 2.8. For matrices $X = [x_{i,j}]$ and $Y = [y_{i,j}]$ in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the $(i, j)^{\text{th}}$ entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Definition 2.9. Let $\mathbb{S}$ be a semiring, not necessary commutative. An operator $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ is called *linear* if $T(\alpha X) = \alpha T(X)$, $T(X\beta) = T(X)\beta$, and $T(X+Y) = T(X) + T(Y)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{S})$, $\alpha, \beta \in \mathbb{S}$.

We say that an operator T *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs, that $(X, Y) \in \mathcal{P}$ implies $(T(X), T(Y)) \in \mathcal{P}$.

An operator T on $\mathbb{M}_{m,n}(\mathbb{S})$ is called a (P, Q, B)*-operator* if there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$, and a matrix $B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $B \geq J$ such that

$$T(X) = P(X \circ B)Q \tag{2.1}$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or, $m = n$ and

$$T(X) = P(X \circ B)^t Q \tag{2.2}$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where X^t denotes the transpose of X. Operators of the form (2.1) are called *non-transposing* (P, Q, B)*-operators*; operators of the form (2.2) are *transposing* (P, Q, B)*-operators.*

An operator T is called a (U, V)*-operator* if there exist invertible matrices $U \in \mathbb{M}_m(\mathbb{S})$ and $V \in \mathbb{M}_n(\mathbb{S})$ such that

$$T(X) = UXV \tag{2.3}$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or, $m = n$ and

$$T(X) = UX^tV \tag{2.4}$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Operators of the form (2.3) are called *non-transposing* (U, V)*-operators*; operators of the form (2.4) are *transposing* (U, V)*-operators.*

Lemma 2.10. *Let T be a (P, Q, B)-operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$. If $\mathbb{S}$ is commutative, then T is a (U, V)-operator.*

Proof. Since T is a (P, Q, B)-operator, so there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = P(X \circ B)Q$, or $m = n$ and $T(X) = P(X \circ B)^t Q$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Since $mc(B) = 1$, so it follows from (1.1) that $rank(B) = 1$, equivalently, there exist vectors $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{S}^m$ and $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{S}^n$ such that $B = \mathbf{d}^t\mathbf{e}$. Since $b_{i,j}$ are units, d_i and e_j are invertible elements in $\mathbb{S}$ for all $(i, j) \in \Delta_{m,n}$. Let $D = diag(d_1, \ldots, d_m) \in \mathbb{M}_m(\mathbb{S})$ and $E = diag(e_1, \ldots, e_n) \in \mathbb{M}_n(\mathbb{S})$ be diagonal matrices. Since $\mathbb{S}$ is commutative, it is straightforward to check that $X \circ B = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. For the case of $T(X) = P(X \circ B)Q$, if we let $U = PD$ and $V = EQ$, then $T(X) = UXV$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. If T is of the form $T(X) = P(X \circ B)^t Q$, then $U = PE$ and $V = DQ$ shows that $T(X) = UX^tV$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus the Lemma follows. ∎

Theorem 2.11. [2, Theorem 2.14] *Let $\mathbb{S}$ be an antinegative semiring without zero divisors and $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ be a linear operator. Then the following are equivalent:*

(1) *T is bijective.*

(2) *T is surjective.*

(3) *There exists a permutation σ on $\Delta_{m,n}$ and units $b_{i,j} \in \mathcal{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$.*

Proof. That 1) implies 2) and 3) implies 1) is straight forward. The fact that the entries $b_{i,j} \in \mathcal{Z}(\mathbb{S})$ follows immediately from the linearity of T. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i, j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indices (r, s) such that $X = x_{r,s}E_{r,s} + X'$ where (r, s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r, s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$ then $T(X) = 0$

which contradicts with the assumption $T(X) = E_{i,j}$. Since $\mathbb{S}$ is antinegative without zero divisors it follows that

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}$$

Hence, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$ and $T(E_{r,s}) \neq O$ by the above. Therefore, $T(E_{r,s}) \leq E_{i,j}$.

Let $P_{i,j} = \{E_{r,s} \mid T(E_{r,s}) \leq E_{i,j}\}$. By the above $P_{i,j} \neq \Phi$ for all (i,j). By its definition $P_{i,j} \cap P_{u,v} = \Phi$ whenever $(i,j) \neq (u,v)$. That is $\{P_{i,j}\}$ is a set of mn nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that $\mid P_{i,j} \mid = 1$ for all (i,j). Necessarily, for each pair (r,s) there is a unique pair (i,j) such that $T(E_{r,s}) = b_{r,s}E_{i,j}$. That is there is some permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that for some scalars $b_{i,j}$, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. We now only need to show that the $b_{i,j}$ are all units. Since T is surjective and $T(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$. That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is a unit. ■

Lemma 2.12. [2, Lemma 2.16] *Let $\mathbb{S}$ be an antinegative semiring without zero divisors, $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on $\Delta_{m,n}$, and $b_{i,j} \in \mathcal{Z}(\mathbb{S})$ are nonzero elements. Then T is a (P,Q,B)-operator.*

Proof. Since no combination of a rows and b columns can dominate J where $a + b = m$ unless $b = 0$ (or if $m = n$, if $a = 0$) we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = Pb_{i,j}E_{i,j}Q = P(E_{i,j} \circ B)Q$, or, if $m = n$, $T(E_{i,j}) = Pb_{i,j}E_{j,i}Q = P(E_{i,j} \circ B)^t Q$ where $B = (b_{i,j})$ is defined by the action of T on the cells. $\blacksquare$

Remark 2.13. One can easily check that if $m = 1$ or $n = 1$ then all operators under consideration are (P, Q, B)-operators, if $m = n = 1$ then all operators under consideration are (P, P^t, B)-operators.

Henceforth we will always assume that $m, n \geq 2$.

Lemma 2.14. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ with $mc(B) = 1$. If all entries of B are units in $\mathcal{Z}(\mathbb{S})$, then $mc(X) = mc(P(X \circ B)Q)$ for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Let X be any matrix in $\mathbb{M}_{m,n}(\mathbb{S})$. Obviously, $mc(X) = mc(XQ)$ for all permutation matrix $Q \in \mathbb{M}_n(\mathbb{S})$. Let P be any permutation matrix in $\mathbb{M}_n(\mathbb{S})$. Then $mc(X) = mc((P)^t PXQ) \leq mc(PXQ) \leq mc(XQ) = mc(X)$, and hence $mc(X) = mc(PXQ)$ for all for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$. Thus, we suffice to claim that $mc(X) = mc(X \circ B)$.

Since $mc(B) = 1$, so there exist the k^{th} column $\mathbf{b_k}$ of $B = [\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}]$ such that $B = \mathbf{b_k}[\alpha_1, \ldots, \alpha_{k-1}, 1, \alpha_{k+1}, \ldots, \alpha_n]$ where α_i are units. Thus, for any matrix $X = [\mathbf{x_1},\ \mathbf{x_2}, \ldots,\ \mathbf{x_n}] \in \mathbb{M}_{\mathbf{m,n}}(\mathbb{S})$, we have $X \circ B = [\mathbf{x_1} \circ \mathbf{b_k}\alpha_1,\ \mathbf{x_2} \circ \mathbf{b_k}\alpha_2, \ldots, \mathbf{x_n} \circ \mathbf{b_k}\alpha_n] = [\mathbf{b_k}\alpha_1 \circ \mathbf{x_1}, \mathbf{b_k}\alpha_2 \circ \mathbf{x_2}, \ldots, \mathbf{b_k}\alpha_n \circ \mathbf{x_n}] = [\alpha_1(\mathbf{x_1} \circ \mathbf{b_k}),\ \alpha_2(\mathbf{x_2} \circ \mathbf{b_k}), \ldots,\ \alpha_n(\mathbf{x_n} \circ \mathbf{b_k})]$.

Thus the Lemma follows. $\blacksquare$

Let $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a matrix in $\mathbb{M}_{2,1}(\mathbb{Z}^+)$. Then we have that $mc(X) = 1$, but $mc(X^t) = 2$. Thus, in general, it is not true that for a matrix $X \in \mathbb{M}_{m,n}(\mathbb{S})$, $mc(X) = 1$ if and only if $mc(X^t) = 1$. But the following is obvious.

Lemma 2.15. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$, whose all entries are units in $\mathcal{Z}(\mathbb{S})$. Then $mc(B) = 1$ if and only if $mc(B^t) = 1$.*

Remark 2.16. Let

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

be a matrix in $\mathbb{M}_4(\mathbb{S})$. Then we can easily show that the first three rows (respectively, four columns) are linearly independent. Thus we have $mc(\Omega) = 4$ and $mc(\Omega^t) = 3$.

Now we consider the following sets of matrices that arise as extremal cases in the inequalities listed in Theorem 2.5.

$\mathcal{A}_1(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = n\}$;
$\mathcal{A}_2(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = 1\}$;
$\mathcal{A}_3(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = |\rho(X) - \rho(Y)|\}$;
$\mathcal{M}_1(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = mc(Y)\}$;
$\mathcal{M}_2(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = 0\}$;
$\mathcal{M}_3(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(X) + mr(Y) > n \text{ and } mc(XY) = 1\}$;
$\mathcal{M}_4(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = \rho(X) + \rho(Y) - n\}$.

In the following sections, we characterize the linear operators that preserve the sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$.

3 Extreme preservers of maximal column rank inequalities of matrix sums

3.1 Linear operators that preserve extreme set $\mathcal{A}_1(\mathbb{S})$

In this section, we investigate the linear operators that preserve the set $\mathcal{A}_1(\mathbb{S})$.

Definition 3.1. We say that $\mathbb{M}_{m,n}(\mathbb{S})$ is *fully maximal* if for each $k \leq \min\{m, n\}$, $\mathbb{M}_{m-k,n-k}(\mathbb{S})$ contains a matrix of maximal column rank $n - k$.

If $m \geq n$, then we can easily show that $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal. But, for $m < n$, $\mathbb{M}_{m,n}(\mathbb{S})$ may be or not fully maximal according to a given semiring $\mathbb{S}$. For example, $\mathbb{M}_{2,3}(\mathbb{Z}^+)$ is fully maximal, while $\mathbb{M}_{2,3}(\mathbb{B})$ is not.

Recall that

$$\mathcal{A}_1(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = n\}.$$

Lemma 3.2. *Let $\mathbb{M}_{m,n}(\mathbb{S})$ be fully maximal, σ be a permutation of $\Delta_{m,n}$, and T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units in $\mathcal{Z}(\mathbb{S})$. If T preserves $\mathcal{A}_1$, then T preserves lines.*

Proof. Suppose that T^{-1} does not map lines to lines. Then, there are two non collinear cells which are mapped to a line. There are two cases, they are mapped to two weighted cells in a column or two weighted cells in a row.

If two non collinear cells are mapped to two weighted cells in a column, we may assume without loss of generality that $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{2,1}$.

If $n \leq m$ it suffices to consider $A = E_{1,1} + E_{2,2} + \ldots + E_{n,n}$. In this case, $T(A)$ has maximal column rank at most $n - 1$, that is, $(O, A) \in \mathcal{A}_1$, $(O, T(A)) \notin \mathcal{A}_1$, a contradiction.

Let us consider the case $m \leq n$. Since $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal there exists a matrix $A' \in \mathbb{M}_{m-2,n-2}(\mathbb{S})$ such that $mc(A') = n - 2$. Let us choose A' with the minimal number of non-zero entries. Let $A = O_2 \oplus A' \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus $mc(A) = mc(A') = n - 2$. Hence $(E_{1,1} + E_{2,2}, A) \in \mathcal{A}_1$. Since T preserves $\mathcal{A}_1$, it follows that $(b_{1,1}E_{1,1} + b_{2,2}E_{2,1}, T(A)) \in \mathcal{A}_1$, that is, $mc(b_{1,1}E_{1,1} + b_{2,2}E_{2,1} + T(A)) = n$. Therefore $mc(T(A)[1, \dots, m; 3, \dots, n]) \geq n - 2$. Since the column rank of any matrix cannot exceed the number of columns, $mc(T(A)[1, \dots, m; 3, \dots, n]) = n - 2$.

Further, $|T(A)[1, \dots, m; 3, \dots, n]| < |A| = |A'|$ since T transforms cells to weighted cells and at least one cell has to be mapped into the second column. Thus we can have an $(m-2) \times (n-2)$ submatrix of $T(A)[1, \dots, m; 3, \dots, n]$ whose column rank is $n - 2$ and the number of whose nonzero entries are less than that of A'. This contradicts the choice of A'.

If two non-collinear cells are mapped to two cells in a row, we may assume without loss of generality that $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{1,2}$. In this case, by considering the matrices $b_{1,1}^{-1}E_{1,1} + b_{2,2}^{-1}E_{2,2}$ and A chosen above, the result follows. Thus, T preserves lines. ■

Theorem 3.3. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m = n \geq 4$. If $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal, then T preserves $\mathcal{A}_1$ if and only if T is a non-transposing (P, Q, B)-operator, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$.*

Proof. By Lemma 2.14, we have that all non-transposing (P, Q, B)-operators with $mc(B) = 1$ preserves $\mathcal{A}_1$.

Suppose that T preserves $\mathcal{A}_1$. By Lemma 3.2 we have that T preserves lines and by applying Theorem 2.11 to Lemma 2.12, we have that T is a (P, Q, B)-operator.

Suppose that $mc(B) \geq 2$. Without loss of generality we may assume that the first two rows and columns of B are linearly independent. Since $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal, there exists a matrix $Y' \in \mathbb{M}_{m-2,n-2}(\mathbb{S})$ such that $mc(Y') = n - 2$. Consider matrices $X = \sum_{i=1}^{m}(b_{i,1}^{-1}E_{i,1} + b_{i,2}^{-1}E_{i,2})$ and $Y = O_2 \oplus Y'$ in $\mathbb{M}_{m,n}(\mathbb{S})$. Then all columns of $X + Y$ are linearly independent and hence $(X, Y) \in \mathcal{A}_1$. But the first two columns of $T(X)+T(Y)$ are equal and hence $mc(T(X),T(Y)) \leq n-1$, a contradiction. Thus $mc(B)=1$.

Since all non-transposing (P,Q,B)-operators with $mc(B)=1$ preserves $\mathcal{A}_1$, it only remains to show that if $m=n$ then the transposition does not preserve $\mathcal{A}_1$. Let $A = \begin{bmatrix} \Omega & O \\ O & I_{n-4} \end{bmatrix}$. Then, by Remark 2.16, we have that $mc(A)=n$ and $mc(A^t)=n-1$, so that $(A,O) \in \mathcal{A}_1$ while $(A^t,O) \notin \mathcal{A}_1$. Thus T is a non-transposing (P,Q,B)-operators with $mc(B)=1$. ■

Corollary 3.4. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m=n \geq 4$, and $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal. If $\mathbb{S}$ is commutative, then T preserves $\mathcal{A}_1$ (if and) only if T is a non-transposing (U,V)-operator.*

Proof. Suppose T preserves $\mathcal{A}_1$. By Theorem 3.2, T is a non-transposing (P,Q,B)-operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $mc(B)=1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$. Since $\mathbb{S}$ is commutative, it follows from Lemma 2.10 that T is a non-transposing (U,V)-operator. ■

3.2 Linear operators that preserve extreme set $\mathcal{A}_2(\mathbb{S})$

Recall that

$$\mathcal{A}_2(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = 1\}.$$

Theorem 3.5. *If T is a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserves $\mathcal{A}_2$, then T is a (P, Q, B)-operator, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$. In particular, if $\mathbb{S}$ is commutative, then T is a (U, V)-operator.*

Proof. Suppose that T does not preserve lines. Then, without loss of generality, we may assume that either $T(E_{1,1}+E_{1,2}) = b_{1,1}E_{1,1}+b_{1,2}E_{2,2}$ or $T(E_{1,1}+E_{2,1}) = b_{1,1}E_{1,1} + b_{2,1}E_{2,2}$. In either cases, let $Y = O$ and X be either $E_{1,1} + E_{1,2}$ or $E_{1,1} + E_{2,1}$, so that $(X, Y) \in \mathcal{A}_2$ while $(T(X), T(Y)) \notin \mathcal{A}_2$, a contradiction. Thus T preserves lines.

By applying Theorem 2.11 to Lemma 2.12 we have that T is a (P, Q, B)-operator.

Suppose that $mc(B) \geq 2$, T preserves $\mathcal{A}_2$. Since $mc(T(J)) = mc(B)$, we have $(J, O) \in \mathcal{A}_2$ while $(T(J), T(O)) \notin \mathcal{A}_2$, a contradiction.

By Lemma 2.10, Since S is commutative, T is a (U, V)-operator. ∎

In general, the converse of Theorem 3.5 may be true or not according to a given semiring $\mathbb{S}$. Obviously, by Lemma 2.14, all non-transposing (P, Q, B)-operators with $mc(B) = 1$ (all entries of B are units in $\mathcal{Z}(\mathbb{S})$) preserve $\mathcal{A}_2$. But the following Examples show that transposing (P, Q, B)-operators may or not preserve $\mathcal{A}_2$ according to given semirings $\mathbb{S}$.

Example 3.6. Consider the semiring $\mathbb{Z}^+$ of all nonnegative integers. Let

$$X = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \oplus O_{n-2} \in \mathbb{M}_n(\mathbb{Z}^+).$$

Then we can easily show that $(X, O) \in \mathcal{A}_2$, while $(X^t, O^t) \notin \mathcal{A}_2$. So, the converse of Theorem 3.5 is not true in this case.

Example 3.7. Consider the Boolean semiring $\mathbb{B} = \{0, 1\}$. Then it is straightforward that for a matrix $A \in \mathbb{M}_n(\mathbb{B})$, $mc(A) = 1$ if and only if all non-zero columns of A are the same. Thus all non-zero rows of A are the same and $mc(A^t) = 1$. That is, for any permutation matrices $P, Q \in \mathbb{M}_n(\mathbb{B})$, we have that $mc(A) = 1$ if and only if $mc(PA^tQ) = 1$. This shows that the converse of Theorem 3.5 is true in this case.

3.3 Linear operators that preserve extreme set $\mathcal{A}_3(\mathbb{S})$

Recall that for $\mathbb{S} \subseteq \mathbb{R}^+$

$$\mathcal{A}_3(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid mc(X + Y) = |\rho(X) - \rho(Y)|\}.$$

Lemma 3.8. *Let $\mathbb{S}$ be any subsemiring of $\mathbb{R}^+$, σ be a permutation of $\Delta_{m,n}$, and T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units and $\min\{m, n\} \geq 3$. If T preserves $\mathcal{A}_3$, then T preserves lines.*

Proof. The sum of three distinct weighted cells has maximal column rank at most 3. Thus $T(E_{1,1} + E_{1,2} + E_{2,1})$ is either a sum of 3 collinear cells, and hence has real rank 1, or is contained in two lines, and hence has real rank 2, or is sum of three cells of maximal column rank 3, and hence has real rank 3.

Now, for $X = E_{1,1} + E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$, we have that $(X, Y) \in \mathcal{A}_3$, and the image of Y is a single weighted cell, and hence $\rho(T(Y)) = 1$. Now, if $\rho(T(X)) = 3$, then $T(X+Y)$ must have maximal column rank 3 or 4, and hence $(T(X), T(Y)) \notin \mathcal{A}_3$, a contradiction. If $\rho(T(X)) = 1$, then $(T(X), T(Y)) \notin \mathcal{A}_3$ since $T(X + Y) \neq O$. Thus $\rho(T(X)) = 2$, and $mc(T(X + Y)) = 1$.

However it is straightforward to see that the sum of four weighted cells has the maximal column rank 1 if and only if they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X + Y)$ is the sum of four weighted cells. These cells do not lie in a line since $\rho(T(X)) = 2$. Thus $T(X + Y)$ must be the sum of four weighted cells which lie in the intersection of two rows and two columns.

Similarly, for any i, j, k, l, $T(E_{i,j} + E_{i,l} + E_{k,j} + E_{k,l})$ in the intersection of two rows and two columns. It follows that any two rows must be mapped into two lines. By the bijectivity of T, if some pair of two rows is mapped into two rows(respectively, columns), then any pair of two rows is mapped into two rows(respectively, columns). Similarly, if some pair of two columns is mapped into two rows(respectively, columns), then any pair of two columns is mapped into two rows(respectively, columns).

Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus, every row is mapped into a line. Similarly for columns. Thus T preserves lines. ■

Theorem 3.9. *Let $\mathbb{S}$ be any subsemiring of $\mathbb{R}^+$, $m \neq n$ or $m = n \geq 4$, and T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$. Then T preserves $\mathcal{A}_3$ if and only if T is a non-transposing (P, Q, B)-operator.*

Proof. By Lemma 2.14, we have that all non-transposing (P, Q, B)-operators with $mc(B) = 1$ preserves $\mathcal{A}_3$.

By applying Lemma 3.7 and Theorem 2.11 to Lemma 2.12, we have that if T preserves $\mathcal{A}_3$, then T is a (P, Q, B)-operator.

Suppose that $mc(B) \geq 2$, $S \subseteq \mathbb{R}^+$ and T preserves $\mathcal{A}_3$. Without loss of generality assume that $n \leq m$. Consider

$$X = \left(\sum_{1 \leq j \leq i \leq n} E_{i,j}\right) \oplus O_{m-n,n}, \quad Y = \left(\sum_{1 \leq i < j \leq n} E_{i,j}\right) \oplus O_{m-n,n}.$$

Then $\rho(X) = n = \rho(T(X))$, $\rho(Y) = n - 1 = \rho(T(Y))$, and $mc(X + Y) = 1 = \rho(X) - \rho(Y)$. That is, $(X, Y) \in \mathcal{A}_3$. But $mc(T(X) + T(Y)) = mc(T(J)) = mc(B) \geq 2 > 1 = \rho(T(X)) - \rho(T(Y))$, a contradiction. Thus $mc(B) = 1$.

Since all non-transposing (P, Q, B)-operators with $mc(B) = 1$ preserves $\mathcal{A}_3$ it remains to show that in the case $m = n$ the operator $X \to X^t$ does not preserve $\mathcal{A}_3$. Let $X = \Omega \oplus O_{n-4}$ and $Y = O_n$. Then $(X, Y) \in \mathcal{A}_3$ while $(X^t, Y^t) \notin \mathcal{A}_3$. ■

4 Extreme preservers of maximal column rank inequalities of matrix products

4.1 Linear operators that preserve extreme set $\mathcal{M}_1(\mathbb{S})$

In this section, we investigate the linear operators that preserve the set $\mathcal{M}_1(\mathbb{S})$.

Recall that

$$\mathcal{M}_1(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = mc(Y)\}.$$

Lemma 4.1. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves $\mathcal{M}_1$. Then T preserves lines.*

Proof. Suppose that T^{-1} does not map columns to lines, without loss of generality, that $T^{-1}(E_{1,1} + E_{2,1}) \geq E_{1,1} + E_{2,2}$. Then $T(I)$ has nonzero entries in at most $n-1$ columns. Suppose $T(I)$ has all zero entries in column j. Then for $X = I$ and $Y = T^{-1}(E_{j,1})$, we have $XY = Y$ however, $T(X)T(Y) = O$. This contradicts the fact that T preserves $\mathcal{M}_1$.

Suppose that T^{-1} does not map rows to lines, without loss of generality, that $T^{-1}(E_{1,1}+E_{1,2}) \geq E_{1,1}+E_{2,2}$. That is $T(E_{1,1}+E_{2,2}) = b_{1,1}E_{1,1}+b_{2,2}E_{1,2}$. Then for $X = b_{1,1}^{-1}E_{1,1}+b_{2,2}^{-1}E_{2,2}+[O_2 \oplus I_{n-2}]$, $T(X)$ has maximal column rank at most $n-1$ since either the first two columns of $T(X)$ are linearly dependent or at least one of the columns from the 3^{rd} through the n^{th} is zero.

Let $Y = T^{-1}(I)$, then we have that $(X, Y) \in \mathcal{M}_1$ since $mc(XZ) = mc(Z)$ for any Z, while $mc(T(X)T(Y)) = mc(T(X)) \leq n-1 < n = mc(I) = mc(T(Y))$ so that $(T(X), T(Y)) \notin \mathcal{M}_1$, a contradiction.

Thus T^{-1} and hence T map lines to lines. ∎

Theorem 4.2. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves $\mathcal{M}_1$. Then T is a non-transposing (P, P^t, B)-operator, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$.*

Proof. By applying Lemma 4.1 and Theorem 2.11 to Lemma 2.12, we have that if T preserves $\mathcal{M}_1$, then T is a (P, Q, B)-operator.

Suppose that $mc(B) \geq 2$, without loss of generality $mc(B[1,2|1,2]) = 2$, and $E_{i,1}QP = E_{i,r}$ for all i. Consider the pair $X = E_{1,1}$, $Y = C_1 + C_2$. Then $XY = E_{1,1} + E_{1,2}$ and $mc(XY) = 1 = mc(Y)$. Thus $(X, Y) \in \mathcal{M}_1$. However, the maximal column rank of $(X \circ B)QP(Y \circ B) = b_{1,r}b_{r,1}E_{1,1} + b_{1,r}b_{r,2}E_{1,2}$ is 1 since $b_{1,r}b_{r,1} = b_{r,1}b_{r,2}^{-1}(b_{1,r}b_{r,2})$ by assumption on $b_{i,j}$ ($b_{i,j}$ are units in $\mathcal{Z}(\mathbb{S})$), that is, the columns of $(X \circ B)QP(Y \circ B)$ are linearly dependent. Thus $mc(T(X)T(Y)) = mc((X \circ B)QP(Y \circ B)) = 1$, $mc(T(Y)) = mc(Y \circ B) = mc(B) \geq 2$. Hence $(T(X), T(Y)) \notin \mathcal{M}_1$, a contradiction. Thus $mc(B) = 1$.

To see that the operator $T(X) = P(X \circ B)^t Q$ does not preserve $\mathcal{M}_1$, it suffices to consider $T_0(X) = X^t D$, where $D = QP$, since a similarity and a Hadamard product with a matrix of maximal column rank 1 and invertible entries preserve $\mathcal{M}_1$. Let

$$X = (D^{-1})^t \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \oplus I_{n-4} \right) \text{ and } Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \oplus I_{n-4}.$$

Then $(X, Y) \in \mathcal{M}_1$ while $(X^t D, Y^t D) \notin \mathcal{M}_1$.

It remains to prove that $Q = P^t$. Assume that $QP \neq I$, and that $X \to (QP)X$ transforms the r^{th} row into the t^{th} row for some $r \neq t$. We consider the matrix $X = \sum\limits_{i \neq t} E_{i,i}$, $Y = E_{r,r}$. Then $(X, Y) \in \mathcal{M}_1$, while for certain invertible elements $b_{i,i} \in \mathcal{Z}(\mathbb{S})$ we have that $T(X)T(Y) = P(X \circ B)QP(Y \circ B)Q = P(\sum\limits_{i \neq t} b_{i,i}E_{i,i})(b_{r,r}E_{t,t})Q = O$. Thus $(T(X), T(Y)) \notin \mathcal{M}_1$, a contradiction.

Hence $Q = P^t$. ∎

Corollary 4.3. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ with $n \geq 4$. If $\mathbb{S}$ is commutative and $1 + 1 \neq 1$, then T preserves $\mathcal{M}_1$ (if and) only if there exist an invertible matrix U and an invertible element α such that $T(X) = \alpha UXU^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Suppose T preserves $\mathcal{M}_1$. By Theorem 4.2, T is a non-transposing (P, P^t, B)-operator, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$. That is, $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$. In the proof of Lemma 2.10, there exist invertible diagonal matrices D and E in $\mathbb{M}_n(\mathbb{S})$ such that $X \circ B = DXE$ and hence $T(X) = PDXEP^t$. Let us show that ED is an invertible scalar matrix.

Define $L(X) = (EP^t)T(X)(EP^t)^{-1} = EDX$ for all $X \in \mathbb{M}_n(\mathbb{S})$. Since T preserves $\mathcal{M}_1$ if and only if L does, it suffice to consider $L(X) = EDX$. Let $G = ED$. Then $G = diag(g_1, \cdots, g_n)$ is an invertible diagonal matrix. Assume that $g_1 \neq g_2$. Consider matrices

$$A = \begin{bmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.1}$$

Let $X = A \oplus O_{n-4}$ and $Y = G^{-1}(B \oplus O_{n-4})$. Since all columns of A are linearly independent, it follows that $mc(A) = mc(X) = mc(L(X)) = 4$ and $mc(B) = mc(Y) = mc(L(Y)) = 2$. Furthermore,

$$XY = \begin{bmatrix} 4g_2^{-1} & 4g_2^{-1} & g_3^{-1} + g_4^{-1} & g_3^{-1} + g_4^{-1} \\ 4g_1^{-1} & 4g_1^{-1} & g_3^{-1} + g_4^{-1} & g_3^{-1} + g_4^{-1} \\ g_1^{-1} + g_2^{-1} & g_1^{-1} + g_2^{-1} & g_4^{-1} & g_4^{-1} \\ g_1^{-1} + g_2^{-1} & g_1^{-1} + g_2^{-1} & g_3^{-1} & g_3^{-1} \end{bmatrix} \oplus O_{n-4}$$

has the maximal column rank at most 2. If $mc(XY) = 1$, then we can easily show that $g_1 = g_2$, a contradiction. Thus $mc(XY) = 2$. That is $(X, Y) \in \mathcal{M}_1$. But

$$L(X)L(Y) = G\left(\begin{bmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \oplus O_{n-4}\right)$$

has the maximal column rank 1 and hence $(L(X), L(Y)) \notin \mathcal{M}_1$. This contradiction shows that $g_1 = g_2$. Similarly, if we consider a matrix $A' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}$, then the parallel argument shows that $g_3 = g_4$. Generally, if $n \geq 5$, then we can split zero block into two parts and take $X' = O_r \oplus A \oplus O_{n-r-4}$ or $X' = O_r \oplus A' \oplus O_{n-r-4}$ for appropriate r. Therefore we have that G is an invertible scalar matrix. That is, $G = ED = \alpha I$ for some invertible element α, equivalently $E = \alpha D^{-1}$. If we let $U = PD$, then $T(X) = P(DXE)P^t = \alpha(PD)X(PD)^{-1} = \alpha UXU^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{S})$. Thus the result follows. ■

4.2 Linear operators that preserve extreme set $\mathcal{M}_2$

Recall that

$$\mathcal{M}_2(\mathbb{S}) = \{(X,Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = 0\}.$$

Lemma 4.4. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves $\mathcal{M}_2$, then T maps columns to columns and rows to rows.*

Proof. Suppose that T does not map columns to columns. Say $T(C_j)$ is not a column. Then $T(J\backslash C_j)$ has no zero column. Thus $(J\backslash C_j, E_{j,j}) \in \mathcal{M}_2$, while $(T(J\backslash C_j), T(E_{j,j})) \notin \mathcal{M}_2$, a contradiction.

Suppose that T does not preserve rows. Say $T(R_i)$ is not a row. Then $T(J\backslash R_i)$ has no zero row. Thus $(E_{i,i}, J\backslash R_i) \in \mathcal{M}_2$, while $(T(E_{i,i}), T(J\backslash R_i)) \notin \mathcal{M}_2$, a contradiction.

Hence T maps columns to columns and rows to rows. ■

Theorem 4.5. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. Then T preserves $\mathcal{M}_2$ if and only if T is a non-transposing (P, P^t, B)-operator, where all entries of B are units in $\mathcal{Z}(\mathbb{S})$.*

Proof. By applying Lemma 4.4 and Theorem 2.11 to Lemma 2.12, we have that if T preserves $\mathcal{M}_2$, then T is a (P,Q,B)-operator.

Since T maps columns to columns, T is clearly a non-transposing (P,Q,B)-operator. Since T is surjective, and hence bijective by Theorem 2.11 we have that every entries in B are invertible.

We now only need show that $Q = P^t$. If not, say $QPE_{r,s} = E_{t,s}$ with $t \neq r$. Then $(E_{t,t}, E_{r,s}) \in \mathcal{M}_2$. However,

$$T(E_{t,t})T(E_{r,s}) = Pb_{t,t}E_{t,t}QPb_{r,s}E_{r,s}Q = b_{t,t}b_{r,s}P(E_{t,t}E_{t,s})Q \neq O$$

so that $(T(E_{t,t}), T(E_{r,s})) \notin \mathcal{M}_2$, a contradiction. Thus $Q = P^t$.

The converse is easily established. ■

4.3 Linear operators that preserve extreme set $\mathcal{M}_3$

Recall that

$$\mathcal{M}_3(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(X) + mc(Y) > n \text{ and } mc(XY) = 1\}.$$

Lemma 4.6. *Let T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves $\mathcal{M}_3$, then T preserves lines.*

Proof. Recall that if $(X, Y) \in \mathcal{M}_3$ then $mc(X) + mc(Y) > n$. We assume that there exists indices $i, j, k, l, i \neq k, j \neq l$ such that nonzero entries of $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line.

Let $T(E_{i,j}) = b_{i,j}E_{s,t}$. Then either $T(E_{k,l}) = b_{k,l}E_{s,t'}$ or $T(E_{k,l}) = b_{k,l}E_{s',t}$. In both cases $mc(T(E_{i,j} + E_{k,l})) = 1$. Let $Y' \in \mathbb{M}_n(\mathbb{S})$ be a matrix such that $Y' + E_{i,j} + E_{k,l}$ is a permutation matrix. We consider $X = E_{i,j} + E_{k,l}$, $Y = Y' + E_{k,l}$. Then $XY = E_{k,l'}$ for some l' and $(X, Y) \in \mathcal{M}_3$. However, since $mc(T(X)) = 1$ in either case, and $mc(T(Y)) \leq n-1$, $mc(T(X)) + mc(T(Y)) \leq n$.

Finally, we have that $(T(X), T(Y)) \notin \mathcal{M}_3$, a contradiction. ■

Theorem 4.7. *Let $n \geq 3$ and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves $\mathcal{M}_3$. Then T is a non-transposing (P, P^t, B)-operator, where $mc(B) = 1$ and all entries of B are units in $\mathcal{Z}(\mathbb{S})$.*

Proof. By applying Lemma 4.6 and Theorem 2.11 to Lemma 2.12, we have that if T preserves $\mathcal{M}_3$, then T is a (P, Q, B)-operator.

Suppose that $mc(B) \geq 2$, without loss of generality $mc(B[1, 2|1, 2]) = 2$, and $E_{i,1}QP = E_{i,r}$, $E_{i,2}QP = E_{i,s}$ for all i. Consider the pair $X = C_1 + C_2$, $Y = I$. Then $(X, Y) \in \mathcal{M}_3$ while $(T(X), T(Y)) \notin \mathcal{M}_3$, a contradiction. Thus $mc(B) = 1$.

To see that the operator $T(X) = P(X \circ B)^t Q$ does not preserve $\mathcal{M}_3$, it suffices to consider $T_0(X) = X^t D$, where $D = QP$, since a similarity and a Hadamard product with a matrix of maximal column rank 1 and invertible entries preserve $\mathcal{M}_3$.

Let

$$X = (D^{-1})^t \begin{bmatrix} O & I_2 \\ O & O \end{bmatrix} \textit{and } Y = \begin{bmatrix} I_{n-1} & O \\ O & O \end{bmatrix}.$$

Then $(X, Y) \in \mathcal{M}_3$ while $(X^tD, Y^tD) \notin \mathcal{M}_3$. This proves that T is a non-transposing (P, Q, B)-operator.

Let us check that $Q = P^t$. Assume that $QP \neq I$, and that $X \to (QP)X$ transforms the p^{th} row into the s^{th} row and r^{th} row into t^{th} row with $r \neq s, t$. These exist since $n \geq 3$. We consider the matrix $X = \sum\limits_{i \neq r} E_{i,i}$, $Y = E_{p,p} + E_{r,r}$. Then $(X, Y) \in \mathcal{M}_3$. And we have that $mc(T(X)) + mc(T(Y)) = n + 1 > n$ and

$$T(X)T(Y) = P(X \circ B)QP(Y \circ B)Q = P(\sum_{i \neq r} b_{i,i}E_{i,i})(b_{p,p}E_{s,p} + b_{r,r}E_{t,r})Q.$$

Thus $mc((T(X)T(Y)) = 2$, that is, $(T(X), T(Y)) \notin \mathcal{M}_3$, a contradiction.

Hence $Q = P^t$. ■

Corollary 4.8. *Let $\mathbb{S} = \mathbb{B},\ \mathbb{Z}^+$ and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$ with $n \geq 3$. Then T preserves $\mathcal{M}_3$ if and only if there is a permutation matrix $P \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Suppose T preserves $\mathcal{M}_3$. By Theorem 4.7, T is a non-transposing (P, P^t, B)-operator, where all entries of B are invertible. Note that if $\mathbb{S} = \mathbb{B},\ \mathbb{Z}^+$, 1 is the only invertible element in $\mathbb{S}$, and hence $B = J$. Thus, there exists a permutation matrix $P \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

The converse is easily established. ■

4.4 Linear operators that preserve extreme set $\mathcal{M}_4$

Recall that

$$\mathcal{M}_4(\mathbb{S}) = \{(X,Y) \in \mathbb{M}_n(\mathbb{S})^2 \mid mc(XY) = \rho(X) + \rho(Y) - n\}.$$

Lemma 4.9. *Let $\mathbb{S}$ be any subsemiring of $\mathbb{R}^+$, σ be a permutation of Δ_n, and T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i,j) \in \Delta_n$, where all $b_{i,j}$ are units. If T preserves $\mathcal{M}_4$, then T preserves lines.*

Proof. If T does not preserve lines, then there exist indices $i,j,k,l, i \neq k, j \neq l$ such that nonzero entries of $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line. Let $X' \in \mathbb{M}_n(\mathbb{S})$ be a matrix such that $X' + E_{i,j} + E_{k,l}$ is a permutation matrix.

We consider $X = X' + E_{i,j} + E_{k,l}$. Then $(X,O) \in \mathcal{M}_4$. However, $mc(T(X)) \leq n-1$, $\rho(T(X)) \leq n-1$ since either $T(X)$ has a zero column or $T(X)$ has two proportional columns since $b_{i,j}$ is invertible. Thus $(T(X),O) \notin \mathcal{M}_4$, a contradiction. ∎

Theorem 4.10. *Let $\mathbb{S}$ be a subsemiring of $\mathbb{R}^+$, and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$. If T preserves $\mathcal{M}_4$, then T is a non-transposing (P,P^t,B)-operator, where $mc(B) = 1$ and all entries of B are units.*

Proof. By applying Lemma 4.9 and Theorem 2.11 to Lemma 2.12, we have that if T preserves $\mathcal{M}_4$, then T is a (P,Q,B)-operator.

Let us check that $Q = P^t$. Assume that $QP \neq I$, and that $X \to (QP)X$ transforms the r^{th} row into the t^{th} row with $r \neq t$. We consider the matrix $X = \sum_{i\neq r} E_{i,i}$, $Y = E_{r,r}$. Then $(X,Y) \in \mathcal{M}_4$, and for certain nonzero $b_{i,i} \in \mathbb{S}$, $T(X)T(Y) = P(X \circ B)QP(Y \circ B)Q = P(\sum_{i\neq r} b_{i,i}E_{i,i})(b_{r,r}E_{t,r})Q \neq O$, that is, $(T(X),T(Y)) \notin \mathcal{M}_4$, a contradiction. Thus $Q = P^t$.

Suppose that $mc(B) \geq 2$, without loss of generality $mc(B[1,2|1,2]) = 2$. Let

$$X = \begin{bmatrix} b_{1,1}^{-1} & b_{1,2}^{-1} \\ b_{2,1}^{-1} & b_{2,2}^{-1} \end{bmatrix} \oplus I_{n-2}.$$

Then $mc(X) = mc(X^2) = n$. Note that from the invertibility of $b_{i,j}$ it follows that $\rho(X) = n$. Indeed, if $b_{i,1}^{-1} = \lambda b_{i,2}^{-1}(i = 1, 2)$, for some $\lambda \in \mathbb{R}^+$, then $\lambda = b_{i,1}^{-1} b_{i,2} \in \mathbb{S}$ which contradicts $mc(B[1, 2|1, 2]) = 2$. Thus $mc(X^2) = 2\rho(X) - n$ or $(X, X) \in \mathcal{M}_4$. But $mc(X \circ B) = mc((X \circ B)^2) = \rho(X \circ B) = n - 1$, and hence $mc((X \circ B)^2) > 2\rho(X \circ B) - n$, so that $(T(X), O) \notin \mathcal{M}_4$, a contradiction. Hence $mc(B) = 1$.

Let

$$X = (D^{-1})^t \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus I_{n-2} \right) \textit{and} \;\; Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus I_{n-2},$$

where $D = QP$. Then $(X, Y) \in \mathcal{M}_4$ while $(X^t D, Y^t D) \notin \mathcal{M}_4$. This proves that T is a non-transposing (P, Q, B)-operator.

Therefore T is a non-transposing (P, P^t, B)-operator, where $mc(B) = 1$. ■

Corollary 4.11. *Let $\mathbb{S}$ be a subsemiring of $\mathbb{R}^+$, and T be a surjective linear operator on $\mathbb{M}_n(\mathbb{S})$, where $n \geq 4$. Then T preserves $\mathcal{M}_4$ if and only if there is an invertible matrix U and an invertible elements α such that $T(X) = \alpha U X U^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Suppose T preserves $\mathcal{M}_4$. By Theorem 4.10, T is a non-transposing (P, P^t, B)-operator, where $mc(B) = 1$ and all entries of B are units; $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$. In the proof of Lemma 2.10, there exist invertible diagonal matrices D and E in $\mathbb{M}_n(\mathbb{S})$ such that $X \circ B = DXE$ and hence that $T(X) = PDXEP^t$. Let us show that ED is an invertible scalar matrix. Similar to the proof of Corollary 4.3, we suffice to consider $L(X) = EDX$ for all $X \in \mathbb{M}_n(\mathbb{S})$. Let $G = ED$. Then $G = diag(g_1, \cdots, g_n)$ is an invertible diagonal matrix. Suppose G is not a scalar matrix. As in Corollary 4.3, we lose no generality in assuming that $g_1 \neq g_2$. Let A and B matrices in (4.1). Let

$$X = A \oplus I_{n-4}, \quad Y = B \oplus I_{n-4}.$$

Then

$$XY = \begin{bmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \oplus I_{n-4}$$

so that $\rho(X) = n-1 = \rho(L(X))$, $\rho(Y) = n-2 = \rho(L(Y))$, and $mc(XY) = n-3$. Thus $(X,Y) \in \mathcal{M}_4$. But

$$L(X)L(Y) = G\left(\begin{bmatrix} 4g_2 & 4g_2 & g_3+g_4 & g_3+g_4 \\ 4g_1 & 4g_1 & g_3+g_4 & g_3+g_4 \\ g_1+g_2 & g_1+g_2 & g_4 & g_4 \\ g_1+g_2 & g_1+g_2 & g_3 & g_3 \end{bmatrix} \oplus I_{n-4}\right)$$

so that $mc(L(X)L(Y)) = n-2$ because $g_1 \neq g_2$. Thus $(L(X), L(Y)) \notin \mathcal{M}_4$, a contradiction. Hence $G = ED = \alpha I$ for some invertible element α. If $U = PD$, then $T(X) = \alpha UXU^{-1}$.

The converse is immediate. ∎

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