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Linear Operators that Preserve Commuting Pairs of Nonnegative Integer Matrices

濟州大學校 大學院

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Linear Operators that Preserve Commuting Pairs of Nonnegative Integer Matrices

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Contents

<**Abstract**>

Linear operators that preserve commuting pairs of nonnegative integer matrices

There are many papers on linear operators that preserve commuting pairs of matrices. We have studied linear operators over Boolean matrices and fuzzy matrices. They gave us the motivation to the research on commuting pairs preservers of matrices over nonnegative integers. Recently, Beasley, Pullman and Song obtained characterizations of the linear operators that strongly preserve and preserve commuting pairs of Boolean matrices, fuzzy matrices and max algebra matrices. In this thesis, we extended their results to the matrices over nonnegative integers. Namely we characterize the linear operators that preserve commuting pairs of matrices over nonnegative integers.

1 Introduction

Partly because of their association with nonnegative real matrices, Boolean matrices [$(0,1)$-matrices with the usual arithmetirc, except $1+1=1$] have been the subject of research by many authors. In 1982, Kim [7] published a compendium of results on the theory and applications of Boolean matrices.

Often, parallels are sought for results known for field-valued matrices, see e.g, deCaen and Gregory [5], Rao and Rao [9, 10], Richman and Schneider [11], Beasley and Pullman [2, 3].

The set of commuting pairs of matrices, $\mathcal{C}$, is the set of (unordered) pairs of matrices (X, Y) such that $XY = YX$. The linear operator T is said to *strongly preserve* $\mathcal{C}$ when T(X)T(Y)=T(Y)T(X) if and only if $XY = YX$.

In 1976 Watkins [12] proved that if $n \geq 4$, $\mathcal{M}$ is the set of $n \times n$ matrices over an algebraically closed field of characteristic 0, and L is a nonsingular linear operator on $\mathcal{M}$ which preserves commuting pairs, then there exists an invertible matrix S in $\mathcal{M}$, a nonzero scalar c, and a linear functional f such that either $\mathrm{L}(X) = cSXS^{-1} + f(X)I$ or $\mathrm{L}(X) = cSX^tS^{-1} + f(X)I$, for all X in $\mathcal{M}$. In 1978, Beasley [1] extended this to the case $n = 3$. Also in [1], Beasley showed that the same characterization holds if $n \geq 3$ and L strongly preserves commuting pairs. The real symmetric and complex Hermitian cases were first investigated by Chan and Lim [4] in 1982; the same results were established as in the general case, with the exception that the invertible matrix must be orthogonal or unitary. Further extensions and generalizations to more general fields were obtained by Radjavi [8] and Choi, Jafarian, and Radjavi [6]. Song and et al obtained characterizations of the linear operators that preserve the commutativity of matrices over nonnegative reals [13] and general Boolean algebras [15].

Here we investigate the set of linear operators on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserve the set of pairs of commuting matrices, where $\mathbb{Z}^+$ is the nonnegative part of the ring of integers $\mathbb{Z}$.

We obtain characterizations of linear operators that preserve commuting pairs of nonnegative integer matrices. In Chapter 2, we introduce most of the definitions, notations, and well - known facts. In chapter 3, we study linear operators that strongly preserve commuting pairs of Boolean matrices. In Chapter 4, we give characterizations of linear operators that preserve commuting pairs of nonnegative integer matrices as following;

Theorem 4.3. Let T be a linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$. Then T is a surjective linear operator which preserves pairs of commuting matrices if and only if there exists an invertible matrix $U \in \mathcal{M}_n(\mathbb{Z}^+)$ such that either

(1) $\mathrm{T}(X) = UXU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$, or

(2) $\mathrm{T}(X) = UX^tU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

2 Definitions and Preliminaries

Let $\mathbb{B} = \{0, 1\}$ be the set with the two operations, addition(+) and multiplication($\cdot$) such that

(1) $0 + 0 = 0$, $0 \cdot 0 = 0$.

(2) $0 + 1 = 1 + 0 = 1$, $0 \cdot 1 = 1 \cdot 0 = 0$.

(3) $1 + 1 =1$, $1 \cdot 1 = 1$.

Then $\mathbb{B}$ is called a *Boolean algebra.* A matrix with entries in $\mathbb{B}$ is called a *Boolean matrix.* We let $\mathcal{M}_{m,n}$ denote the set of all $m \times n$ Boolean matrices. The $n \times n$ identity matrix I_n and the $m \times n$ zero matrix $O_{m,n}$ are defined as for a field. The $m \times n$ matrix all of whose entries are zero except its (i,j)th, which is 1, is denoted $E_{i,j}$. We call $E_{i,j}$ a *cell.* We denote the $m \times n$ matrix all of whose entries are 1 by $J_{m,n}$. We omit the subscripts on I, O, and J when they are implied by the context.

Example. If A and B are $n \times n$ Boolean matrices, then $A + I$ commutes with $B + I$ whenever A commutes with B. On the other hand, when $E_{1,1}$ does not commute with J. Therefore $X \to X + I$ preserves commuting pairs of Boolean matrices, but not strongly.

If A and B are in $\mathcal{M}(=\mathcal{M}_{m,n})$, we say B *dominates* A(written B$\geq$A or A$\leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i,j. This provides a reflexive, transitive relation on $\mathcal{M}$.

Linearity of transformations is defined as for vector spaces over fields. A linear transformation on $\mathcal{M}$ is completely determined by its behavior on the set of cells. The number of nonzero entries in a matrix A is denoted $|A|$. A matrix S having at least one nonzero off-diagonal entry is a *line matrix* if all its nonzero entries lie on a line (a row or a column); so $1\leq |S| \leq n$. If the nonzero entries in S are all in a row, we call S a *row matrix* and S^t a *column matrix*. We use R_i (respectively, C_i) to denote the row matrix (respectively, column matrix), respectively with all entries in the ith row (respectively, column) equal 1. We say that cells E and F are *collinear* if there is a line matrix L such that $L \geq E + F$. When X and Y are in $\mathcal{M}$, we define $X \setminus Y$ to be the matrix Z such that $z_{i,j} = 1$ if and only if $x_{i,j} = 1$ and $y_{i,j} = 0$. For example, the matrix in $\mathcal{M}_{n,n}$ having all off-diagonal entries 1 and all diagonal entries 0 is denoted K_n. Thus, $K_n = J \setminus I$. A linear operator T on $\mathcal{M}$ is said to be *nonsingular* if $\mathrm{T}(X)=O$ implies that $X = O$. A nonsingular linear operator on $\mathcal{M}$ need not be invertible. If U is any matrix whose first column has all entries 1, then $X \to XU$ is nonsingular but never invertible, unless $m = n = 1$. Similarly, a matrix A is said to be *nonsingular* if $A\mathbf{x} = 0$ implies that $\mathbf{x}= 0$ ($\mathbf{x}$ a column vector). If A has a nonzero entry in each column, then A is nonsingular. Also, when $m = n$, the only invertible matrices are permutation matrices. Therefore, many nonsingular Boolean matrices are not invertible. We let $\mathcal{C}(A)$ denote the commutator semigroup of A, i.e., $\mathcal{C}(A) = \{\mathrm{X} \in \mathcal{M} \mid XA = AX\}$. Then $\mathcal{C}(J)$ consists of O and the matrices X such that both X and X^t are nonsingular. Let $\mathbb{S}$ denote the set of all symmetric matrices in $\mathcal{M}_{n,n}$. We define a *digon matrix* to be the sum of a cell and its transpose. A *star matrix* is the sum of a line matrix and its transpose. Clearly all digon matrices and all star matrices are symmetric. Let $\hat{\mathcal{C}}(J)$ denote the subsemigroup of $\mathcal{C}(J)$ which lies in $\mathbb{S}_n$, that is, $\hat{\mathcal{C}}(J)$ is the commutator of J in $\mathbb{S}$. Then $\hat{\mathcal{C}}(J)$ is the set of all symmetric nonsingular matrices together with O.

Evidently, the following operations strongly preserve the set of commuting pairs of matrices;

(a) transposition ($X \to X^t$) ;

(b) similarity ($X \to SXS^{-1}$ for some fixed invertible matrix S).

3 Linear operators that strongly preserve commuting pairs of Boolean matrices

In this section, we obtain the characterizations of linear operators that strongly preserve commuting pairs of Boolean matrices. In Lemma 3.1 through 3.8, we let $\mathcal{M} = \mathcal{M}_{n,n}$, and T be a linear operator on $\mathcal{M}$

LEMMA 3.1. *If $A \in \mathcal{C}(J)$ is nonzero and $B \geq A$, then $B \in \mathcal{C}(J)$.*

Proof. Notice that $A \in \mathcal{C}(J)$ if and only if both A and A^t are nonsingular, A has no zero row or column if and only if A and A^t are nonsingular. Therefore, if $A \in \mathcal{C}(J)$ and $B \geq A$, then B has no zero row or column. Hence $B \in \mathcal{C}(J)$.

■

LEMMA 3.2. *If* T *strongly preserves $\mathcal{C}(J)$, then* T *is bijective on the set of cells in $\mathcal{M}$.*

Proof. First, we show that T is nonsingular. We may assume that $n > 1$. If $\mathrm{T}(E) = O$ for some cell E, let M be a minimal matrix in $\mathcal{C}(J)$ dominating E, that is, $|M| \leq |X|$ for all $X \in \mathcal{C}(J)$ with $E \leq X$. Such a matrix exists because $J \in \mathcal{C}(J)$. Moreover, $M \neq E$, as $E \notin \mathcal{C}(J)$ because E is singular. Then $\mathrm{T}(M)=\mathrm{T}(E + M \setminus E)=\mathrm{T}(M \setminus E)$, contrary to the fact that $M \in \mathcal{C}(J)$ and $M \setminus E \notin \mathcal{C}(J)$.

Since $\mathcal{M}$ is finite, there exists some integer $p > 0$ such that T^p is idempotent. Let $Q = \mathrm{T}^p$. Then Q preserves both $\mathcal{C}(J)$ and $\mathcal{M} \setminus \mathcal{C}(J)$, and Q is nonsingular.

Suppose E and F are cells and $E \leq Q(F)$. If $E \neq F$, then there is a line matrix L such that $L \geq E$ and $L \ngeq F$. Let $N = J \setminus L$, then $N + E \in \mathcal{C}\ (J)$. but $N + F \notin \mathcal{C}(J)$, as the former has no zero row or column and the latter does. We have $Q(N + E) = Q(N) + Q(E) \leq Q(N + F)$. Since $N + E \in \mathcal{C}(J)$, we have $Q(N+E) \in \mathcal{C}(J)$, and hence $Q(N + F) \in \mathcal{C}(J)$ by Lemma 3.1. But $N + F \notin \mathcal{C}(J)$. This contradicts the fact that Q preserves $\mathcal{M} \setminus \mathcal{C}(J)$. Thus $E = F$, and hence Q is the identity on the cells of $\mathcal{M}$. Therefore T is bijective on the cells of $\mathcal{M}$. ■

LEMMA 3.3. *If* T *strongly preserves* $\mathcal{C}(J)$, *then* T *preserves the set of line matrices.*

Proof. Suppose M is a line matrix. If E and F are noncollinear cells and $E+F \leq \mathrm{T}(M)$, choose a permutation matrix $P \geq E + F$ and let $X = \mathrm{T}^{-1}\ (P \setminus (E + F))$. Then $X + M$ has a zero row or a zero column, and so $X + M \notin \mathcal{C}\ (J)$, even though $\mathrm{T}(X + M) \geq P \in \mathcal{C}(J)$. $\mathrm{T}(X + M) \in \mathcal{C}(J)$. By Lemma 3.1 a contradiction to the fact that T preserves $\mathcal{M} \setminus \mathcal{C}(J)$. ■

LEMMA 3.4. *If* T *strongly preserves* $\mathcal{C}(J)$, *then either*

(a) T *maps row matrices to row matrices and column matrices to column matrices ; or*

(b) T *maps row matrices to column matrices and column matrices to row matrices.*

Proof. According to Lemma 3.3, $\mathrm{T}(R_1)$ is a row matrix R_i, or a column matrix C_i, for some i. Suppose that $\mathrm{T}(R_1) = R_i$. Select j distinct from i, and suppose that the line matrix $\mathrm{T}(R_j)$ is the column matrix C_l.

Then $\mid \mathrm{T}(R_1+R_j) \mid = \mid R_i+C_l \mid <$ 2n. while $\mid R_1+R_j \mid = 2n$, contradicting the bijectivity of T . Hence T maps row matrices to row matrices and (a) holds. A similar argument establishes (b) when $\mathrm{T}(R_1)$ is a column matrix. ■

Lemma 3.5. *If* T *strongly preserves* $\mathcal{C}(J)$*, then there are permutation matrices P and Q such that either (i)* $\mathrm{T}(X) = PXQ$ *for all* $X \in \mathcal{M}$ *or (ii)* $\mathrm{T}(X) = PX^tQ$ *for all* $X \in \mathcal{M}$*.*

Proof. By Lemma 3.4 either (a) T maps row matrices to row matrices and column matrices to column matrices, or (b)T maps row matrices to column matrices and column matrices to row matrices. Since T is bijective, no two lines can be mapped to the same line. Let P be the permutation matrix that corresponds to the mapping T induces between the row indices and the row [column] indices, and let Q be the permutation matrix that corresponds to the mapping T induces between the column indices and the column [row] indices, according as $(a)[(b)]$ holds. Obviously (i) follows when (a) holds and (ii) follows when (b) holds. ■

Theorem 3.1. *A linear operator* T *on* $\mathcal{M}$ *strongly preserves commuting pairs if and only if there exists a permutation matrix P such that either* (a) $\mathrm{T}(X) = PXP^t$ *for all* $X \in \mathcal{M}$ *or* (b) $\mathrm{T}(X) = PX^tP^t$ *for all* $X \in \mathcal{M}$*.*

Proof. We only need to prove the necessity, and we may assume that $n >$1. Suppose that T strongly preserves commuting pairs. We first show that $\mathrm{T}(J) = J$. Suppose $\mathrm{T}(J) = Y$, and choose $X = [x_{i,j}] \in \mathcal{M}$ such that $\mathrm{T}(X) = Y$ and $\mid X \mid \leq$ |A | for all A with $\mathrm{T}(A) = Y$. Since $\mathrm{T}(X)$=$\mathrm{T}(J)$, $\mathcal{C}(X)$ =$\mathcal{C}(J)$.

Suppose $x_{1,1} = 1$. Since $JP = PJ$, $PJP^t = J$ for any permutation matrix P. It follows from $\mathcal{C}(X) = \mathcal{C}(J)$ that $PXP^t = X$ for any permutation matrix P. Thus $x_{i,i} = 1$ for all $i = 1, 2, \ldots, n$. Now, X has a nonzero off-diagonal entry; Otherwise $X \leq I$ and thus $\mathcal{C}(I) = \mathcal{C}(J)$, which is impossible for n >1. Without loss of generality we may assume that $x_{1,2} = 1$. We note that $PX = XP$ for any permutation P. In particular, if P fixes the first row[column], it follows that the first row of X has all entries equal 1. Also, $P_iX = XP_i$, where P_i is the permutation matrix that interchanges the first row [column] with the ith and fixes the rest, $2 \leq i \leq n$. It follows that $X = J$.

Now suppose $x_{1,1} = 0$. An argument similar to the above shows that $x_{i,j} = 0$ if and only if $i = j$. Now J, and hence X, commutes with $I + E_{1,2}$. However, $X(I + E_{1,2})$ has $(2, 2)$ entry equal to 1, while $(I + E_{1,2})X$ has $(2, 2)$ entry equal to 0, a contradiction.

Thus $\mathrm{T}(J) = J$, and hence T strongly preserves $\mathcal{C}(J)$. Let P_n denote the set of $n \times n$ permutation matrices. Then if case (i) of Lemma 3.5 holds, we have $\mathrm{T}(\mathsf{P}_n) = \mathsf{P}_n$ and $\mathrm{T}(I) = PQ$. Then PQ commutes with every member of P_n and hence $PQ = I$. A similar argument holds if case (ii) of Lemma 3.5 holds. ■

In the following lemmas, T is a linear operator on $\mathbb{S}$

LEMMA 3.6. *If $A \in \hat{C}(J)$, $B \in \mathbb{S}$, and $B \geq A$, then $B \in \hat{C}(J)$.*

Proof. Notice that $A \in \hat{C}(J)$ if and only if both A and A^t are symmetric nonsingular, A has no zero row or column if and only if A and A^t are nonsingular. Therefore, if $A \in$

$\hat{C}(J)$ and $B \geq A$, then B has no zero row or column. Hence $B \in \hat{C}(J)$. ■

LEMMA 3.7. *If* T *strongly preserve* $\hat{C}(J)$*, then* T *is bijective on the set of digon matrices of* $\mathcal{M}$*, where* $n > 2$*.*

Proof. First, we show that T is nonsingular. We may assume that $n > 2$. If $\mathrm{T}(E) = O$ for some digon matrix E, let M be a minimal matrix in $\hat{C}(J)$ dominating E, that is, $|M| \leq |X|$ for all $X \in \hat{C}(J)$ with $E \leq X$. Such a matrix exists because $J \in \hat{C}(J)$. Moreover, $M \neq E$, as $E \notin \hat{C}(J)$ because E is singular. Then $\mathrm{T}(M)$=$\mathrm{T}(E+M \setminus E)$=$\mathrm{T}(M \setminus E)$, contrary to the fact that $M \in \hat{C}(J)$ and $M \setminus E \notin \hat{C}(J)$.

Since $\mathcal{M}_{n,n}$ is finite, there exists some integer p >0 such that T^p is idempotent. Let Q =T^p. Then Q preserves both $\hat{C}(J)$ and $\mathcal{M}_{n,n} \setminus \hat{C}(J)$, and Q is nonsingular. Suppose E and F are digon matrices and $E \leq Q(F)$. If $E \neq F$, then there is a star matrix S such that $S \geq E$ and $S \ngeq$F. Let $N = J \setminus S$, then $N + E \in \hat{C}(J)$. but $N + F \notin \hat{C}(J)$, as the former has no zero row or column and the latter does. We have $Q(N + E) = Q(N) + Q(E) \leq Q(N + F)$. Since $N + E \in \hat{C}(J)$, we have $Q(N + E) \in \hat{C}(J)$, and hence $Q(N + F) \in \hat{C}(J)$ by Lemma3.6. But $N + F \notin \hat{C}(J)$. This contradicts the fact that Q preserves $\mathcal{M}_{n,n} \setminus \hat{C}(J)$. Thus $E = F$,and hence Q is the identity on the digon matrices of $\mathcal{M}_{n,n}$. Therefore T is bijective on the digon matrices of $\mathcal{M}_{n,n}$. ■

LEMMA 3.8. *If* T *strongly preserves* $\hat{C}(J)$*, then* T *preserves the set of star matrices.*

Proof. Suppose S is a star matrix. If E and F are digon matrices not dominated by a star matrix S and $E+F \leq \mathrm{T}(S)$, Choose a permutation matrix $P \geq E+F$ and let X=T^{-1} $(P \setminus (E+F))$. Then $X+S$ has a zero row or column, so $X+S \notin \hat{C}(J)$, even though $\mathrm{T}(X+S) \geq P \in \hat{C}$ (J) by Lemma 3.6 contradicting the fact that T preserves $\mathcal{M} \setminus \hat{C}$ (J). ■

THEOREM 3.2. *A linear operator* T *on* $\mathbb{S}$ *strongly preserves commuting pairs if and only if there exists a permutation matrix* P *such that either (a)* $\mathrm{T}(X) = PXP^t$ *for all* $X \in \mathbb{S}$ *or (b)*$\mathrm{T}(X) = PX^tP^t$ *for all* $X \in \mathbb{S}$.

Proof. Let σ be the map of $\{1, 2, \ldots, n\}$ to itself defined by $\sigma(i) = j$ if and only if the maximal star matrix on row and column i is mapped to one on row and column j. By Lemma 3.7, σ is one-to-one, and hence onto. Let P be the permutation matrix corresponding to σ. Then the results follow. ■

4 Linear operators that preserve commuting pairs of nonnegative integer matrices.

In this section, we characterize the linear operators that preserve commuting pairs of nonnegative integer matrices. We let $\mathcal{M}_n(\mathbb{Z}^+)$ denote the set of $n \times n$ matrices over $\mathbb{Z}^+$. A mapping T: $\mathcal{M}_n(\mathbb{Z}^+) \longrightarrow \mathcal{M}_n(\mathbb{Z}^+)$ is called a *linear operator* if $\mathrm{T}(\alpha A + \beta B) = \alpha\ \mathrm{T}(A) + \beta\ \mathrm{T}(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{Z}^+)$ and for all $\alpha, \beta \in \mathbb{Z}^+$. For $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{Z}^+)$, we recall that A *dominates* B, denoted by $A \geq B$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$. Let T be a linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$. If A and B are matrices in $\mathcal{M}_n(\mathbb{Z}^+)$ with $A \leq B$, we can easily show that $\mathrm{T}(A) \leq \mathrm{T}(B)$. Let $\Delta_n = \{(i,j) \,|\, 1 \leq i, j \leq n\}$. Then for any $(i,j) \in \Delta_n$, we recall that $E_{i,j}$ denotes the $n \times n$ matrix whose (i,j)th entry is 1 and other entries are all 0. We call $E_{i,j}$ a *cell.*

LEMMA 4.1. *Let* T: $\mathcal{M}_n(\mathbb{Z}^+) \longrightarrow \mathcal{M}_n(\mathbb{Z}^+)$ *be a linear operator on* $\mathcal{M}_n(\mathbb{Z}^+)$ *. Then the following are equivalent :*

1. T *is bijective.*
2. T *is surjective.*
3. *There exists a permutation* σ *on* Δ_n *such that* $\mathrm{T}(E_{i,j}) = E_{\sigma_{(i,j)}}$

Proof. That 1) implies 2) and 3) implies 1) is straight forward. We now show that 2) implies 3). We assume that T is surjective. Then, for any pair $(i,j) \in \Delta_n$, there exists a matrix $X \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $\mathrm{T}(X) = E_{i,j}$. Clearly

$X \neq O$ by the linearity of T. Thus there is $(r,s) \in \Delta_n$ such that $X = x_{r,s}E_{r,s} + X'$ where (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $\mathrm{T}(E_{r,s}) \neq O$. Since $\mathbb{Z}^+$ has no zero divisors it follows that

$$\mathrm{T}(x_{r,s}E_{r,s}) \leq \mathrm{T}(x_{r,s}E_{r,s}) + \mathrm{T}(X \setminus (x_{r,s}E_{r,s})) = \mathrm{T}(X) = E_{i,j},$$

equivalently,

$$\mathrm{T}(x_{r,s}E_{r,s}) = x_{r,s}\mathrm{T}(E_{r,s}) \leq E_{i,j},$$

and so $\mathrm{T}(E_{r,s}) \leq E_{i,j}$.
It follows from $x_{r,s} \neq 0$ that $\mathrm{T}(E_{r,s}) = b_{r,s}E_{i,j}$ for some nonzero scalar $b_{r,s}$. Let $P_{i,j} = \{E_{r,s} \mid \mathrm{T}(E_{r,s}) \leq E_{i,j}\}$. By the above $P_{i,j} \neq \phi$ for all $(i,j) \in \Delta_n$. By its definition, $P_{i,j} \cap P_{u,v} = \phi$ whenever $(i,j) \neq (u,v)$. That is, $\{P_{i,j}\}$ is the set of n^2 nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that $\mid P_{i,j} \mid = 1$ for all $(i,j) \in \Delta_n$. Necessarily, for each pair (r,s) there is the unique pair (i,j) such that $\mathrm{T}(E_{r,s}) = b_{r,s}E_{i,j}$. Thus, there is some permutation σ on $\{(i,j) \mid i,j = 1,2,\ldots,n\}$ such that $\mathrm{T}(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, for scalars $b_{i,j}$. We now only need to show that $b_{i,j} = 1$, for all i,j. Since T is surjective and $\mathrm{T}(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $\mathrm{T}(\alpha E_{i,j}) = E_{\sigma(i,j)}$. Since T is linear,

$$E_{\sigma(i,j)} = \mathrm{T}(\alpha E_{i,j}) = \alpha\ \mathrm{T}(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)}.$$

That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is unit. Since 1 is the only unit element in $\mathbb{Z}^+$, $b_{i,j} = 1$ for all $(i,j) \in \Delta_n$ ■

We denotes $\mathcal{C}_n(\mathbb{Z}^+)$ as the set of commuting pairs of matrices over $\mathbb{Z}^+$; that is, $\mathcal{C}_n(\mathbb{Z}^+) = \{(A,B) \in \mathcal{M}^2(\mathbb{Z}^+) \mid AB = BA\}$.

Example 4.2 Let A be given in $\mathcal{M}_n(\mathbb{Z}^+)$. Define an operator T on $\mathcal{M}_n(\mathbb{Z}^+)$ by

$$T(X) = \left(\sum_{i,j=1}^{n} x_{i,j}\right) A$$

for all X=$[x_{i,j}] \in \mathcal{M}_n(\mathbb{Z}^+)$. Then we can easily show that T is a linear operator that preserve commuting pairs of matrices, while it does not preserve non-commuting pairs of matrices. ■

Thus, we are interested in linear operators that

$$(T(A), T(B)) \in \mathcal{C}_n(\mathbb{Z}^+) \text{ if and only if } (A, B) \in \mathcal{C}_n(\mathbb{Z}^+).$$

For a matrix $A \in \mathcal{M}_n(\mathbb{Z}^+)$, A is called *invertible* in $\mathcal{M}_n(\mathbb{Z}^+)$ if there exists a matrix $B \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $AB = BA = I_n$. It is well known [2] that all permutation matrices are only invertible matrices in $\mathcal{M}_n(\mathbb{B})$. Using this fact, we can easily show that all permutation matrices are only invertible matrices in $\mathcal{M}_n(\mathbb{Z}^+)$.

Theorem 4.3. *Let* T *be a linear operator on* $\mathcal{M}_n(\mathbb{Z}^+)$. *Then* T *is a surjective linear operator which preserves pairs of commuting matrices if and only if there exists an invertible matrix* $U \in \mathcal{M}_n(\mathbb{Z}^+)$ *such that either*

(1) $\mathrm{T}(X) = UXU^t$ *for all* $X \in \mathcal{M}_n(\mathbb{Z}^+)$, *or*

(2) $\mathrm{T}(X) = UX^tU^t$ *for all* $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

Proof. Let T be a surjective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ that preserves pairs of commuting matrices. By Lemma 4.1, T is bijective and there exists a permutation σ on Δ_n

such that $\mathrm{T}(E_{i,j}) = E_{\sigma(i,j)}$. Note that if $AX = XA$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$, then we have $A = \alpha I_n$ for some $\alpha \in \mathbb{Z}^+$. Thus we have $\mathrm{T}(I_n) = \beta I_n$ for some $\beta \in \mathbb{Z}^+$ because T is bijective. Since T maps a cell onto a cell, $\mathrm{T}(I_n) = I_n$. It follows that there is a permutation γ of $\{1, \ldots, n\}$ such that $T(E_{i,i}) = E_{\gamma(i)\gamma(i)}$ for each $i = 1, \ldots, n$. Define L: $\mathcal{M}_n(\mathbb{Z}^+) \to \mathcal{M}_n(\mathbb{Z}^+)$ by $\mathrm{L}(X) = P\mathrm{T}(X)P^t$, where P is the permutation matrix corresponding to γ so that $\mathrm{L}(E_{i,i}) = E_{i,i}$ for each $i = 1, \ldots, n$. Then we can easily show that L is a bijective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserves pairs of commuting matrices. By Lemma 4.1, L maps a cell onto a cell. Therefore, there exists $(p, q) \in \Delta_n$ such that $\mathrm{L}(E_{r,s}) = E_{p,q}$ for any $(r, s) \in \Delta_n$.

Suppose that $r \neq s$. Since L is bijective, we have $p \neq q$ because $\mathrm{L}(E_{i,i}) = E_{i,i}$ for each $i = 1, \ldots, n$. Assume that $p \neq r$ and $p \neq s$. Then

$$E_{r,s}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{r,s}$$

so that

$$\mathrm{L}(E_{r,s})\mathrm{L}(E_{r,r} + E_{s,s} + E_{p,p}) = \mathrm{L}(E_{r,r} + E_{s,s} + E_{p,p})\mathrm{L}(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{p,q}.$$

It follows that $q = r$ or $q = s$. Since $E_{r,s}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{r,s}$, we have

$$\mathrm{L}(E_{r,s})\mathrm{L}(E_{r,r} + E_{s,s}) = \mathrm{L}(E_{r,r} + E_{s,s})\mathrm{L}(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{p,q}.$$

Since $q = r$ or $q = s$, we have $E_{p,q}(E_{r,r} + E_{s,s}) = E_{p,r}$ or $E_{p,s}$, but $(E_{r,r} + E_{s,s})E_{p,q} = 0$, a contradiction. Hence we have $p = r$ or $p = s$. Similarly we obtain $q = r$ or $q = s$. Therefore we have $\mathrm{L}(E_{r,s}) = E_{r,s}$ or $\mathrm{L}(E_{r,s}) = E_{s,r}$ for each $(r, s) \in \Delta_n$. Suppose that $\mathrm{L}(E_{r,s}) = E_{r,s}$ with $r \neq s$ and $\mathrm{L}(E_{r,t}) = E_{t,r}$ for some $t \neq r, s$. Then we have $\mathrm{L}(E_{s,t} + E_{t,s}) = E_{s,t} + E_{t,s}$. Let $A = E_{r,r} + E_{s,t} + E_{t,s}$ so that $\mathrm{L}(A) = E_{r,r} + E_{s,t} + E_{t,s}$. Then $(E_{r,s} + E_{r,t})A = A(E_{r,s} + E_{r,t})$, and hence

$$\mathrm{L}(E_{r,s} + E_{r,t})\mathrm{L}(A) = \mathrm{L}(A)\mathrm{L}(E_{r,s} + E_{r,t}).$$

But

$$\mathrm{L}(E_{r,s} + E_{r,t})\mathrm{L}(A) = E_{r,t} + E_{t,r},$$

while

$$\mathrm{L}(A)\mathrm{L}(E_{r,s} + E_{r,t}) = E_{r,s} + E_{s,r}.$$

Thus we have $t = s$, a contradiction. It follows that if $\mathrm{L}(E_{i,j}) = E_{i,j}$ for some pair $(i, j) \in \Delta_n$ with $i \neq j$, then $\mathrm{L}(E_{r,s}) = E_{r,s}$ for all pairs $(r, s) \in \Delta_n$. Similarly, if $\mathrm{L}(E_{i,j}) = E_{j,i}$ for some pair $(i, j) \in \Delta_n$ with $i \neq j$, then $\mathrm{L}(E_{r,s}) = E_{s,r}$ for all pairs $(r, s) \in \Delta_n$. We have established that either $\mathrm{L}(X) = X$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$ or $\mathrm{L}(X) = X^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. Therefore $\mathrm{T}(X) = P^tXP$ or $\mathrm{T}(X) = P^tX^tP$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. If $U = P^t$, then we have $\mathrm{T}(X) = UXU^t$ or $\mathrm{T}(X) = UX^tU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

The converse is immediate. ■

Thus we have characterized the linear operators that preserve commuting pairs of matrices over nonnegative integers.

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< 국 문 초 록 >

음이 아닌 정수 행렬들의 교환 쌍을 보존하는 선형연산자들

본 논문에서는 부울 행렬과 실수 상에서의 행렬들의 교환 쌍을 보존하는 선형연산자들의 특성에 관한 기존의 논문 결과가 음이 아닌 정수 행렬의 경우에도 적용할 수 있는가를 고찰하였다. 즉, 비음의 정수에서 원소를 갖는 행렬들 위에서 가환행렬의 짝들을 보존하는 선형연산자의 특성을 밝혔다.

그 결과 음이 아닌 정수 행렬 상에서 선형연산자가 교환행렬의 쌍을 보존하는 전사인 선형연산자일 필요충분조건은 모든 행렬에 대하여 $T(X) = UXU^t$ 또는 $T(X) = UX^tU^t$ 를 만족하는 가역행렬 U가 $M_n(\mathbb{Z}^+)$ 에 존재하는 것임을 밝혔다.

< 감사의 글 >

이렇게 석사학위논문을 쓰게 되어 기쁘게 생각합니다. 먼저, 논문을 쓰기까지 많은 지도, 편달 해주신 송석준 교수님과 석사기간 동안 강의를 해 주신 방교수님, 정교수님, 윤교수님, 유교수님, 논문심사를 해주신 양교수님, 진교수님께 진심어린 감사의 말씀을 드립니다. 그리고 논문 퇴고 시 세심한 관심을 가져 주신 강경태 선생님께도 감사의 말씀을 드립니다. 또, 6개월 동안 동고동락한 동석이형, 같은 곳을 향해 달려가고 있다는 것만으로도, 저에게 많은 힘이 되었습니다. 그리고 가끔씩 커피 친구가 되어준 후배 승빈이게도 고맙다는 말을 전한다. 마지막으로 저를 가장 많이 아끼시고 사랑해 주신 부모님께 고맙다는, 사랑한다는 말을 전하고 싶습니다. 이런 많은 분들이 도움이 있었기에 석사과정을 순조롭게 끝낼 수 있었던 것 같습니다. 모든 분들께 다시 한번 감사하다는 말을 전합니다.